

TWO DIMENSIONAL MONADICITY

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ABSTRACT. The behaviour of limits of weak morphisms in 2-dimensional universal algebra is not 2-categorical in that, to fully express the behaviour that occurs, one needs to be able to quantify over strict morphisms amongst the weaker kinds. \mathcal{F} -categories were introduced to express this interplay between strict and weak morphisms. We express doctrinal adjunction as an \mathcal{F} -categorical lifting property and use this to give monadicity theorems, expressed using the language of \mathcal{F} -categories, that cover each weaker kind of morphism.

1. INTRODUCTION

The category of monoids sits over the category of sets via a forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$. This functor is *monadic* in the sense that it has a left adjoint F and the canonical comparison $E : \mathbf{Mon} \rightarrow \mathbf{Set}^T$ to the category of algebras for the induced monad $T = UF$ is an equivalence of categories. So if you are interested in monoids you can set about proving some theorem about algebras for an abstract monad T and be sure it holds for monoids, or any variety of universal algebra for that matter: this is the categorical approach to universal algebra via monads.

Before going down this path one thing must be established – namely, the monadicity of U . To this end the fundamental theorem is *Beck’s monadicity theorem* [1] which asserts that a functor $U : \mathcal{A} \rightarrow \mathcal{B}$ is monadic just when it admits a left adjoint, is conservative and creates U -absolute coequalisers. What makes the theorem so useful in practice is that the conditions, up to the *existence* of a left adjoint, are cast entirely in terms of the typically simple U – these conditions are clearly met for monoids or indeed any variety (see Section 6.8 of [17]).

Now our interest is not in universal algebra, but in two dimensional universal algebra and 2-monads, and monadicity as appropriate to this setting. What do we mean by the varieties of 2-dimensional universal algebra? Monoidal structure borne by categories provides a basic example: one observes that associated to this notion are several kinds of structure. On the objects front we have at least strict monoidal categories and monoidal categories of interest and between these are strict, strong, lax and colax monoidal functors all commonly arising, between which we have just one kind of monoidal transformation. Restricting ourselves to just one kind of object, let us take the monoidal categories, we still find that we are presented with four connected 2-categories \mathbf{MonCat}_w , where $w \in \{s, p, l, c\}$,

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living over the 2-category of categories \mathbf{Cat} as on the left below.

$$(1) \quad \begin{array}{ccc} \text{MonCat}_s & \longrightarrow & \text{MonCat}_p \\ & \searrow V_s & \downarrow V_p \\ & & \text{Cat} \end{array} \quad \begin{array}{ccc} & \nearrow V_l & \nearrow V_c \\ \text{MonCat}_p & \longrightarrow & \text{MonCat}_l \\ & \searrow V_p & \downarrow V_c \\ & & \text{Cat} \end{array}$$

$$\begin{array}{ccc} \text{T-Alg}_s & \longrightarrow & \text{T-Alg}_p \\ & \searrow U_s & \downarrow U_p \\ & & \text{Cat} \end{array} \quad \begin{array}{ccc} & \nearrow U_l & \nearrow U_c \\ \text{T-Alg}_p & \longrightarrow & \text{T-Alg}_l \\ & \searrow U_p & \downarrow U_c \\ & & \text{Cat} \end{array}$$

The objects in each of these are monoidal categories; the morphisms are respectively strict, strong (or pseudo), lax and colax monoidal functors with monoidal transformations between them in each case. The inclusions witness that strict morphisms can be viewed as pseudomorphisms ($s \leq p$), which can in turn be viewed as lax or colax ($p \leq l$) and ($p \leq c$).

The situation corresponds with that of a 2-monad T based on \mathbf{Cat} , associated with which are several kinds of algebra including strict and pseudo-algebras, although we will only ever consider the *strict algebras*. Between these are strict, pseudo, lax and colax morphisms, with again a single notion of algebra transformation. As before we obtain a diagram of 2-categories¹ and 2-functors over \mathbf{Cat} , as on the right above.

Comparing the diagrams left and right above we see that a monadicity theorem in this setting ought to match each 2-category on the left with the corresponding one on the right in a compatible way. More precisely, there should exist a 2-monad T on \mathbf{Cat} and, for each $w \in \{s, p, l, c\}$, an equivalence of 2-categories $E_w : \text{MonCat}_w \rightarrow \text{T-Alg}_w$ over \mathbf{Cat} , as left below

$$(2) \quad \begin{array}{ccc} \text{MonCat}_w & \xrightarrow{E_w} & \text{T-Alg}_w \\ & \searrow V_w & \nearrow U_w \\ & & \text{Cat} \end{array} \quad \begin{array}{ccc} \text{MonCat}_{w_1} & \xrightarrow{E_{w_1}} & \text{T-Alg}_{w_1} \\ \downarrow & & \downarrow \\ \text{MonCat}_{w_2} & \xrightarrow{E_{w_2}} & \text{T-Alg}_{w_2} \end{array}$$

and these equivalences should be natural in the inclusions $w_1 \leq w_2$ for $w_1, w_2 \in \{s, p, l, c\}$, as on the right.

Now it is well known that such a 2-monad does indeed exist, with, moreover, each comparison $E_w : \text{MonCat}_w \rightarrow \text{T-Alg}_w$ an isomorphism of 2-categories. Likewise many of the other varieties of 2-dimensional universal algebra are monadic in this sense, and with such varieties as the primary object of study the subject of 2-dimensional monad theory was developed, notably in [2], and general theorems proved, such as those enabling one to deduce that each inclusion $\text{MonCat}_s \rightarrow \text{MonCat}_w$ has a left 2-adjoint, or deduce the bicategorical completeness and co-completeness of MonCat_p .

Of course to apply such abstract results one must first establish monadicity. How

¹The objects in each of these 2-categories are the strict algebras.

is this known? Here the subject diverges substantially from the 1-dimensional approach of Beck's theorem. In the standard approach, that of colimit presentations [14][10], one explicitly constructs the intended 2-monad T as an iterated colimit of free ones and then performs lengthy calculations with the universal property of the colimit T to establish monadicity in the sense described for monoidal categories above.

Now although the natural analogue of Beck's theorem has been obtained for *pseudomonads* [16] and pseudoalgebra pseudomorphisms this does not specialise to capture monadicity in the precise sense described above, even when $w = p$. Our objective in the present paper is to give such 2-dimensional monadicity theorems, in which monadicity is recognised not by using an explicit description of a 2-monad, but by analysing the manner in which the varieties of 2-dimensional universal algebra sit over the base 2-category – as in Diagram 1. Apart from characterising such monadic situations our main monadicity results, Theorem 19 through Theorem 21, enable one to establish monadicity when a workable description of a 2-monad is not easily forthcoming.

In seeking to understand monadicity in the above sense the first important observation is that the world of *strict morphisms* is easily understood: one can check that $V_s : \text{MonCat}_s \rightarrow \text{Cat}$ has a left 2-adjoint – say, by an adjoint functor theorem – and then apply the enriched version of Beck's monadicity theorem [3] to establish strict monadicity. Having observed this to be the case the natural question to ask is: *Which properties of the commutative triangle*

$$\begin{array}{ccc} \text{MonCat}_s & \longrightarrow & \text{MonCat}_w \\ & \searrow V_s & \swarrow V_w \\ & \text{Cat} & \end{array}$$

ensure that the canonical isomorphism $E : \text{MonCat}_s \rightarrow \text{T-Alg}_s$ extends to an isomorphism $E_w : \text{MonCat}_w \rightarrow \text{T-Alg}_w$ over the base? We answer this question in Theorem 21 but not using the language of 2-categories. For it turns out that these determining properties are not 2-categorical in nature – they cannot be expressed as properties of the 2-categories or 2-functors in the above diagram considered individually – rather, to express these properties we must be able to *single out strict morphisms amongst each weaker kind*. Consequently we treat the inclusion $\text{MonCat}_s \rightarrow \text{MonCat}_w$ as a single entity, an \mathcal{F} -category MonCat_w , and the above triangle as a single \mathcal{F} -functor $V : \text{MonCat}_w \rightarrow \text{Cat}$.

Let us now give an overview of the paper and of our line of argument. \mathcal{F} -categories were introduced in [15] in order to explain certain relationships between strict and weak morphisms in 2-dimensional universal algebra. We recall some basic facts about \mathcal{F} -categories in Section 2, in particular discussing \mathcal{F} -categories of algebras T-Alg_w for a 2-monad and \mathcal{F} -categories MonCat_w of monoidal categories – we will use monoidal categories as our running example throughout the paper.

Given a strict monoidal functor F and an adjunction $(\epsilon, F \dashv G, \eta)$ the right adjoint G obtains a unique lax monoidal structure such that the adjunction becomes a monoidal adjunction. This is an instance of *doctrinal adjunction*, the main topic of Section 3, and the first key \mathcal{F} -categorical property that we meet. We express

three variants of doctrinal adjunction – w -doctrinal adjunction for $w \in \{l, p, c\}$ – as lifting properties of an \mathcal{F} -functor, so that the case just described asserts that the forgetful \mathcal{F} -functor $V : \mathbb{M}\text{onCat}_l \rightarrow \text{Cat}$ satisfies l -doctrinal adjunction. We define the closely related class $w\text{-Doct}$ of w -doctrinal \mathcal{F} -functors and analyse the relationships between the different notions for $w \in \{l, p, c\}$; each such class of \mathcal{F} -functor is shown to be an orthogonality class in the category of \mathcal{F} -categories.

In the fourth section we turn to the reason \mathcal{F} -categories were introduced in [15] – namely, because of the interplay between strict and weak morphisms, *tight* and *loose*, that occurs when considering limits of weak morphisms in 2-dimensional universal algebra. The crucial limits are \overline{w} -limits of loose morphisms for $w \in \{l, p, c\}$ – after defining these we describe the illuminating case of the colax limit of a lax monoidal functor. We then examine how such limits allow one to represent loose morphisms by *tight spans* – the nature of this representation is analysed in detail.

This analysis allows us to prove the key result of the paper, Theorem 17 of Section 5, an orthogonality result which has nothing to do with 2-monads at all. Its immediate consequence, Corollary 18, ensures that the decomposition in $\mathcal{F}\text{-CAT}$

$$\text{MonCat}_s \xrightarrow{j} \mathbb{M}\text{onCat}_w \xrightarrow{V} \text{Cat} = \text{MonCat}_s \begin{array}{c} \xrightarrow{1} \text{MonCat}_s \xrightarrow{V_s} \text{Cat} \\ \searrow j \downarrow j \text{MonCat}_w \nearrow V_w \end{array}$$

of the forgetful 2-functor $V_s : \text{MonCat}_s \rightarrow \text{Cat}$ is an orthogonal $({}^\perp w\text{-Doct}, w\text{-Doct})$ -decomposition; likewise for a 2-monad T on Cat the \mathcal{F} -category $T\text{-Alg}_w$ is obtained as a $({}^\perp w\text{-Doct}, w\text{-Doct})$ -factorisation of $U_s : T\text{-Alg}_s \rightarrow \text{Cat}$.

Our monadicity results, given in Section 6, use the uniqueness of $({}^\perp w\text{-Doct}, w\text{-Doct})$ -decompositions to extend our understanding of monadicity in the strict setting to cover each weaker kind of morphism. For instance, the isomorphism $E : \text{MonCat}_s \rightarrow T\text{-Alg}_s$ over Cat induces a commuting diagram as on the outside of

$$\begin{array}{ccccc} \text{MonCat}_s & \xrightarrow{j} & \mathbb{M}\text{onCat}_w & \xrightarrow{V} & \text{Cat} \\ E \downarrow & & \vdots E_w \downarrow & & \downarrow 1 \\ T\text{-Alg}_s & \xrightarrow{j} & T\text{-Alg}_w & \xrightarrow{U} & \text{Cat} \end{array}$$

with each horizontal path an orthogonal $({}^\perp w\text{-Doct}, w\text{-Doct})$ -decomposition; now the two outer vertical isomorphisms induce a unique invertible filler $E_w : \text{MonCat}_w \rightarrow T\text{-Alg}_w$, so establishing monadicity in each weaker context. This is the idea behind the main monadicity result, Theorem 21. Naturality in different weaknesses (as in Diagram 2 above) is treated in Theorem 20.

In the seventh and final section we describe examples and applications of our results. We begin by completing the example of monoidal categories before moving on to more complex cases; we conclude in Theorem 23 with an example of the kind of monadicity result that cannot be established using techniques, like presentations, that require explicit knowledge of a 2-monad.

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2. \mathcal{F} -CATEGORIES IN 2-DIMENSIONAL UNIVERSAL ALGEBRA

In this section we recall the notion of \mathcal{F} -category, introduced in [15], and a few basic facts about them.

2.1. \mathcal{F} -categories. An \mathcal{F} -category \mathbb{A} is a 2-category with two kinds of 1-cell: those of the 2-category itself which are called *loose* and a subcategory of *tight* morphisms containing all of the identities. A second perspective is that an \mathcal{F} -category \mathbb{A} is specified by a pair of 2-categories \mathcal{A}_τ and \mathcal{A}_λ connected by a 2-functor

$$j : \mathcal{A}_\tau \rightarrow \mathcal{A}_\lambda$$

which is the *identity on objects, faithful and locally fully faithful*: here \mathcal{A}_λ contains all of the morphisms, which is to say the loose ones, and all 2-cells between them, whilst \mathcal{A}_τ contains the tight morphisms together with all 2-cells between them in \mathcal{A}_λ . Loose morphisms in \mathbb{A} are drawn with wavy arrows $A \rightsquigarrow B$ and tight morphisms with straight arrows $A \rightarrow B$, so that a typical diagram in \mathbb{A} would be

$$\begin{array}{ccc} A & \overset{f}{\rightsquigarrow} & B \\ & \searrow \alpha & \downarrow g \\ & h & C \end{array}$$

The viewpoint of \mathcal{F} -categories as special 2-functors hints at another one: for each pair of objects $A, B \in \mathbb{A}$ the inclusion of hom categories

$$j_{A,B} : \mathcal{A}_\tau(A, B) \rightarrow \mathcal{A}_\lambda(A, B)$$

constitutes an *injective on objects fully faithful functor* and in fact \mathcal{F} -categories are precisely categories enriched in \mathcal{F} , the full subcategory of the arrow category \mathbf{Cat}^2 whose objects are those functors which are both injective on objects and fully faithful. \mathcal{F} is a complete and cocomplete cartesian closed category so that the full theory of enriched categories [8] can be applied to the study of \mathcal{F} -categories.

To begin with we have $\mathcal{F}\text{-CAT}$, the 2-category of \mathcal{F} -categories, \mathcal{F} -functors and \mathcal{F} -natural transformations. An \mathcal{F} -functor $F : \mathbb{A} \rightarrow \mathbb{B}$ consists of a pair of 2-functors $F_\tau : \mathcal{A}_\tau \rightarrow \mathcal{B}_\tau$ and $F_\lambda : \mathcal{A}_\lambda \rightarrow \mathcal{B}_\lambda$ rendering commutative the square

$$\begin{array}{ccc} \mathcal{A}_\tau & \xrightarrow{j_A} & \mathcal{A}_\lambda \\ F_\tau \downarrow & & \downarrow F_\lambda \\ \mathcal{B}_\tau & \xrightarrow{j_B} & \mathcal{B}_\lambda \end{array}$$

This equally amounts to a 2-functor $F_\lambda : \mathcal{A}_\lambda \rightarrow \mathcal{B}_\lambda$ which *preserves tightness*. An \mathcal{F} -natural transformation $\eta : F \rightarrow G$ is a 2-natural transformation $\eta : F_\lambda \rightarrow G_\lambda$ with *tight components*.

2.2. Concepts definable in \mathcal{F} -categories. Any concept definable in a 2-category can certainly be considered in the \mathcal{F} -categorical context; thus adjunctions, equivalences and so on. When we speak of an *adjunction or equivalence in an \mathcal{F} -category* \mathbb{A} we will mean an adjunction or equivalence in its 2-category \mathcal{A}_λ of loose morphisms, though we may speak, for instance, of an adjunction in \mathbb{A} with *tight left adjoint*. Just as a 2-functor preserves adjunctions so too will \mathcal{F} -functors but also adjunctions with tight left adjoint.

2.3. 2-categories as \mathcal{F} -categories. Each 2-category \mathcal{A} may be viewed as an \mathcal{F} -category in two extremal ways: as an \mathcal{F} -category in which only the identities are tight, or as an \mathcal{F} -category in which *all morphisms are tight*, whereupon the induced \mathcal{F} -category has the form

$$1 : \mathcal{A} \rightarrow \mathcal{A}$$

When we view a 2-category \mathcal{A} as an \mathcal{F} -category it will *always be in this second sense* and we again denote it by \mathcal{A} .

With this convention established we can treat 2-categories as special kinds of \mathcal{F} -categories and unambiguously speak of \mathcal{F} -functors $F : \mathcal{A} \rightarrow \mathbb{B}$ from 2-categories to \mathcal{F} -categories, or from \mathcal{F} -categories to 2-categories as in $G : \mathbb{B} \rightarrow \mathcal{C}$. These \mathcal{F} -functors appear as triangles

$$\begin{array}{ccc} & \mathcal{A} & \\ F_\tau \swarrow & & \searrow F_\lambda \\ \mathcal{B}_\tau & \xrightarrow{j_B} & \mathcal{B}_\lambda \end{array} \qquad \begin{array}{ccc} \mathcal{B}_\tau & \xrightarrow{j_A} & \mathcal{B}_\lambda \\ G_\tau \searrow & & \swarrow G_\lambda \\ & \mathcal{C} & \end{array}$$

with an \mathcal{F} -functor between 2-categories just a 2-functor. Observe that each \mathcal{F} -category \mathbb{A} induces an \mathcal{F} -functor from its 2-category of tight morphisms

$$j : \mathcal{A}_\tau \rightarrow \mathbb{A}$$

which is the identity on tight morphisms and $j : \mathcal{A}_\tau \rightarrow \mathcal{A}_\lambda$ on loose ones – we abuse notation by using j in either situation.

2.4. \mathcal{F} -categories of monoidal categories. In two-dimensional universal algebra one encounters morphisms of four different flavours and so \mathcal{F} -categories naturally arise. Here we recall the various \mathcal{F} -categories associated to the notion of monoidal structure – we recall the defining equations for monoidal functors as we will use these later on.

The data for a monoidal category consists of a tuple $\overline{A} = (A, \cdot, i^A, \lambda^A, \rho_l^A, \rho_r^A)$ where we use juxtaposition for the tensor product. A lax monoidal functor $(F, f, f_0) : \overline{A} \rightsquigarrow \overline{B}$ consists of a functor $F : A \rightarrow B$, coherence constraints $f_{a,b} : FaFb \rightarrow F(ab)$ natural in a and b and a comparison $f_0 : i^B \rightarrow Fi^A$, all satisfying the first

three conditions below

$$\begin{array}{ccccc}
 (FaFb)Fc & \xrightarrow{f_{a,b}1} & F(ab)Fc & \xrightarrow{f_{ab,c}} & F((ab)c) & i^B Fa & \xrightarrow{f_0 1} & Fi^A Fa & \xrightarrow{f_{i,a}} & F(i^A.a) \\
 \downarrow \lambda_{Fa,Fb,Fc}^B & & & & F\lambda_{a,b,c}^A \downarrow & \rho_i^B \downarrow & & & & \downarrow F\rho_i^A \\
 Fa(FbFc) & \xrightarrow{1f_{b,c}} & FaF(bc) & \xrightarrow{f_{a,bc}} & F(a(bc)) & Fa & \xlongequal{\quad\quad\quad} & Fa & & Fa
 \end{array}$$

$$\begin{array}{ccc}
 (Fa)i^B \xrightarrow{1f_0} FaFi^A \xrightarrow{f_{a,i}} F(a.i^A) & \left| \right. & FaFb \xrightarrow{\eta_a \eta_b} GaGb \\
 \rho_r^B \downarrow & & f_{a,b} \downarrow \quad \downarrow g_{a,b} \\
 Fa \xlongequal{\quad\quad\quad} Fa & & F(ab) \xrightarrow{\eta_{ab}} G(ab)
 \end{array}
 \quad
 \begin{array}{ccc}
 i^B \xrightarrow{f_0} Fi^A & & \\
 \searrow g_0 & \downarrow \eta_{i^A} & \\
 & Gi^A &
 \end{array}$$

which we call the *associativity*, *left unit* and *right unit* conditions. We call (F, f, f_0) strong or strict monoidal just when the constraints $f_{a,b}$ and f_0 are invertible or identities respectively; reversing these constraints we obtain the notion of a colax monoidal functor. Between lax monoidal functors are monoidal transformations $\eta : (F, f, f_0) \rightarrow (G, g, g_0)$: these are natural transformations $\eta : F \rightarrow G$ satisfying the final two conditions above, which we call the *tensor* and *unit* conditions for a monoidal transformation.

For $w \in \{s, l, p, c\}$ we thus have w -monoidal functors (with p -monoidal meaning strong monoidal). Together with monoidal categories and monoidal transformations these form a 2-category MonCat_w which sits over Cat via a forgetful 2-functor $V_w : \text{MonCat}_w \rightarrow \text{Cat}$. Now strict monoidal functors are strong ($s \leq p$) and strong monoidal functors can be viewed as lax ($p \leq l$) or colax ($p \leq c$); thus whenever $w_1 \leq w_2$ we have an \mathcal{F} -category MonCat_{w_1, w_2} of monoidal categories with tight and loose morphisms the w_1 and w_2 -monoidal functors respectively, as specified by the inclusion $j : \text{MonCat}_{w_1} \rightarrow \text{MonCat}_{w_2}$. Each such \mathcal{F} -category comes equipped with a forgetful \mathcal{F} -functor $V : \text{MonCat}_{w_1, w_2} \rightarrow \text{Cat}$ where $V_\tau = V_{w_1}$ and $V_\lambda = V_{w_2}$ – see the commuting triangles in Diagram 1 of the introduction. Of particular importance will be those \mathcal{F} -categories $\text{MonCat}_{s, w}$ for $s \leq w$, which we denote by MonCat_w .

2.5. \mathcal{F} -categories of algebras. Of prime importance are those \mathcal{F} -categories associated to a 2-monad T on a 2-category \mathcal{C} . Each 2-monad has various associated flavours of algebra and morphism. We will only be interested in *strict algebras* and will call them algebras. Between algebras we have strict, pseudo, lax and colax morphisms – as with monoidal functors we specify these using s, p, l and c .

Drawn in turn from left to right below is the data (f, \bar{f}) for a strict, pseudo, lax or colax morphism of algebras $(f, \bar{f}) : (A, a) \rightsquigarrow (B, b)$

$$\begin{array}{cccc}
 \begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array} &
 \begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & \bar{f} \cong & \downarrow b \\ A & \xrightarrow{f} & B \end{array} &
 \begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & \Downarrow \bar{f} & \downarrow b \\ A & \xrightarrow{f} & B \end{array} &
 \begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & \Uparrow \bar{f} & \downarrow b \\ A & \xrightarrow{f} & B \end{array}
 \end{array}$$

Thus \bar{f} is an identity 2-cell in the first case, invertible in the second, and points into or out of f in the lax or colax cases; in all cases these 2-cells are required

to satisfy two coherence conditions [11] that are automatically satisfied by the identities of the strict case.

There is a single notion of 2-cell between any kind of algebra morphisms; for instance given a pair of lax morphisms $(f, \bar{f}), (g, \bar{g}) : (A, a) \rightsquigarrow (B, b)$ an algebra 2-cell $\alpha : (f, \bar{f}) \Rightarrow (g, \bar{g})$ is a 2-cell $\alpha : f \Rightarrow g$ satisfying

$$\begin{array}{ccc}
 \begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 a \downarrow & \bar{f} \Downarrow & \downarrow b \\
 A & \xrightarrow{f} & B \\
 & \alpha \Downarrow & \\
 & \xrightarrow{g} &
 \end{array}
 & = &
 \begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 a \downarrow & T\alpha \Downarrow & \downarrow b \\
 A & \xrightarrow{Tg} & B \\
 & \Downarrow \bar{g} & \\
 & \xrightarrow{g} &
 \end{array}
 \end{array}$$

whilst the equation in the colax case looks like the lax case with the directions reversed.

Algebras, w -algebra morphisms and transformations live in 2-categories $\mathbf{T}\text{-Alg}_w$ for $w \in \{s, p, l, c\}$, each of which comes equipped with an evident forgetful 2-functor to the base, which we always denote by $U_w : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$. Each strict morphism is a pseudo morphism ($s \leq p$), and each pseudomorphism can be viewed either as lax ($p \leq l$) or colax ($p \leq c$) so that for each pair $w_1, w_2 \in \{s, p, l, c\}$ satisfying $w_1 \leq w_2$ we have an \mathcal{F} -category $\mathbf{T}\text{-Alg}_{w_1, w_2}$ with tight morphisms the w_1 -morphisms, and loose morphisms the w_2 -morphisms. Each of these comes equipped with a forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_{w_1, w_2} \rightarrow \mathcal{C}$ (so $U_\tau = U_{w_1}$ and $U_\lambda = U_{w_2}$) – see the commuting triangles of Diagram 1 of the introduction.

Of particular importance are those \mathcal{F} -categories $\mathbf{T}\text{-Alg}_{s, w}$ whose tight morphisms are the strict ones, and following [15] we abbreviate these by $\mathbf{T}\text{-Alg}_w$. As well as using $j : \mathbf{T}\text{-Alg}_s \rightarrow \mathbf{T}\text{-Alg}_w$ for the defining inclusion and $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ for the forgetful \mathcal{F} -functor, we occasionally use j_w or U_w if we are in the presence of multiple j 's or U 's.

2.6. Duality for algebras. Colax algebra morphisms are lax algebra morphisms with 2-cells reversed. This statement can be made precise using the covariant duality 2-functor $(-)^{co} : 2\text{-CAT} \rightarrow 2\text{-CAT}$ which takes a 2-category \mathcal{C} to the 2-category \mathcal{C}^{co} with the same underlying category but with 2-cells reversed. Likewise it takes a 2-monad $T : \mathcal{C} \rightarrow \mathcal{C}$ to a 2-monad $T^{co} : \mathcal{C}^{co} \rightarrow \mathcal{C}^{co}$ and one then has, as noted in [6], an equality $\mathbf{T}^{co}\text{-Alg}_l = \mathbf{T}\text{-Alg}_c^{co}$ which restricts to $\mathbf{T}^{co}\text{-Alg}_s = \mathbf{T}\text{-Alg}_s^{co}$. The $(-)^{co}$ duality naturally extends to a 2-functor

$$(-)^{co} : \mathcal{F}\text{-CAT} \rightarrow \mathcal{F}\text{-CAT}$$

under whose action we have that $\mathbf{T}\text{-Alg}_c^{co} = \mathbf{T}^{co}\text{-Alg}_l$ and moreover that

$$U^{co} : \mathbf{T}\text{-Alg}_c^{co} \rightarrow \mathcal{C}^{co} \quad \text{equals} \quad U : \mathbf{T}^{co}\text{-Alg}_l \rightarrow \mathcal{C}^{co} .$$

A consequence of this duality is that each theorem about lax morphisms has a dual version concerning colax morphisms. Indeed all of our definitions and results in the colax case will be dual to those in the lax setting – though we will *state*

these results for colax morphisms it will always suffice to *prove* results only in the lax setting.

2.7. Equivalence of \mathcal{F} -categories. Our monadicity theorems in Section 6 will assert that certain \mathcal{F} -categories are *equivalent* to \mathcal{F} -categories of algebras for a 2-monad. By an equivalence of \mathcal{F} -categories we mean an equivalence in the 2-category of \mathcal{F} -categories $\mathcal{F}\text{-CAT}$, which is to say an equivalence of \mathcal{V} -categories for $\mathcal{V} = \mathcal{F}$.

Recall from [8] that a \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is an equivalence just when it is *essentially surjective on objects* and *fully faithful* in the enriched sense. The notion of essential surjectivity is cast in terms of the underlying category \mathcal{A}_0 of a \mathcal{V} -category: this has the same objects as \mathcal{A} and homs given by $\mathcal{A}_0(A, B) = \mathcal{V}(I, \mathcal{A}(A, B))$ where I is the monoidal unit. Now F is said to be essentially surjective just when the underlying functor $F_0 : \mathcal{A}_0 \rightarrow \mathcal{B}_0$ is, and fully faithful when $F_{A,B} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(FA, FB)$ is an isomorphism in \mathcal{V} for each pair $A, B \in \mathcal{A}$. In the case of the cartesian closed \mathcal{F} the underlying category \mathbb{A}_0 of an \mathcal{F} -category \mathbb{A} is just the underlying category of \mathcal{A}_τ . Therefore an \mathcal{F} -functor $F : \mathbb{A} \rightarrow \mathbb{B}$ is essentially surjective on objects in the enriched sense just when F_τ is essentially surjective – note that this also implies the weaker statement that F_λ is essentially surjective on objects. Enriched fully faithfulness says that each $F_{A,B} : \mathbb{A}(A, B) \rightarrow \mathbb{B}(FA, FB)$ is an isomorphism in \mathcal{F} ; in other words both $F_{\tau A, B} : \mathcal{A}_\tau(A, B) \rightarrow \mathcal{B}_\tau(FA, FB)$ and $F_{\lambda A, B} : \mathcal{A}_\lambda(A, B) \rightarrow \mathcal{B}_\lambda(FA, FB)$ are isomorphisms of categories.

We conclude that F is an equivalence of \mathcal{F} -categories just when both 2-functors F_τ and F_λ are essentially surjective on objects and 2-fully faithful, which is to say that both F_τ and F_λ are 2-equivalences, or equivalences of 2-categories.

3. DOCTRINAL ADJUNCTION AND \mathcal{F} -CATEGORICAL LIFTING PROPERTIES

If a strict monoidal functor has a right adjoint that right adjoint admits a unique lax monoidal structure such that the adjunction lifts to a monoidal adjunction. This is an instance of *doctrinal adjunction* – the topic of the present section. We begin by recalling Kelly’s treatment of doctrinal adjunction in the setting of 2-monads, recasting the notion in \mathcal{F} -categorical terms, so that the above special case becomes the assertion that the forgetful \mathcal{F} -functor $V : \text{MonCat}_l \rightarrow \text{Cat}$ satisfies *l-doctrinal adjunction* – we treat each of $w \in \{l, p, c\}$. In Section 3.2 we define the closely related notion of a *w-doctrinal \mathcal{F} -functor* before showing that the *w-doctrinal \mathcal{F} -functors* form an orthogonality class in $\mathcal{F}\text{-CAT}$.

3.1. Doctrinal adjunction \mathcal{F} -categorically. Doctrinal adjunction was first studied in Kelly’s paper [6] of the same name. Motivated by the example of adjunctions between monoidal categories amongst others, all known to be describable using 2-monads via clubs [7] or other techniques, he gave his treatment in the setting of 2-dimensional monad theory – let us now recall the relevant aspects of this. Given T -algebras (A, a) and (B, b) and a morphism $f : A \rightarrow B$ together with an adjunction $(\epsilon, f \dashv g, \eta)$ in the base, his Theorem 1.2 asserts that there is a bijection between colax algebra morphisms of the form $(f, \bar{f}) : (A, a) \rightsquigarrow (B, b)$ and lax morphisms of the form $(g, \bar{g}) : (B, b) \rightsquigarrow (A, a)$. The structure 2-cells $\bar{f} : f.a \Rightarrow b.Tf$

and $\bar{g} : a.Tg \Rightarrow g.b$ are expressed in terms of one another as mates as below

$$\begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 \downarrow 1 & \nearrow T\eta \Rightarrow & \downarrow b \\
 TA & & B \\
 \downarrow a & \searrow \bar{g} & \downarrow 1 \\
 A & \xrightarrow{f} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 TB & \xrightarrow{Tg} & TA \\
 \downarrow 1 & \nearrow T\epsilon \Leftarrow & \downarrow a \\
 TB & & A \\
 \downarrow b & \searrow \bar{f} & \downarrow 1 \\
 B & \xrightarrow{g} & A
 \end{array}$$

Since lax and colax morphisms cannot be composed this relationship cannot be expressed 2-categorically or indeed \mathcal{F} -categorically – it can be captured using *double categories* as in Example 5.4 of [18]. However if we start with (f, \bar{f}) a pseudomorphism then it does live in the same 2-category as the resultant lax morphism $(g, \bar{g}) : (B, b) \rightsquigarrow (A, a)$. Moreover it is shown in Proposition 1.3 of [6] that the unit and counit η and ϵ then become algebra 2-cells $\eta : 1 \Rightarrow (g, \bar{g}) \circ (f, \bar{f})$ and $\epsilon : (f, \bar{f}) \circ (g, \bar{g}) \Rightarrow 1$ in $\mathbf{T-Alg}_1$ and so yield an adjunction $(\epsilon, (f, \bar{f}) \dashv (g, \bar{g}), \eta)$ in $\mathbf{T-Alg}_1$.

Dually, if (g, \bar{g}) were a pseudomorphism then once f is equipped with the corresponding colax structure (f, \bar{f}) the adjunction $(\epsilon, f \dashv g, \eta)$ lifts to an adjunction $(\epsilon, (f, \bar{f}) \dashv (g, \bar{g}), \eta)$ in the 2-category $\mathbf{T-Alg}_c$.

The invertibility of \bar{f} does not imply the invertibility of its mate \bar{g} , or vice-versa. However, if both η and ϵ are invertible, then \bar{f} is invertible just when \bar{g} is, which is to say that if $(\epsilon, f \dashv g, \eta)$ is an adjoint equivalence and either f or g admits the structure of a pseudomorphism the adjoint equivalence lifts to an adjoint equivalence in $\mathbf{T-Alg}_p$.

Let us now abstract these lifting properties of adjunctions and adjoint equivalences into properties of the forgetful \mathcal{F} -functors $U : \mathbf{T-Alg}_{w_1, w_2} \rightarrow \mathcal{C}$. Given an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ and an adjunction $(\epsilon, f \dashv g, \eta)$ in \mathbb{B} a *lifting of this adjunction along* W is an adjunction $(\epsilon', f' \dashv g', \eta')$ in \mathbb{A} such that $Wf' = f, Wg' = g, W\epsilon' = \epsilon$ and $W\eta' = \eta$. We will likewise speak of liftings of adjoint equivalences along W . Now an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ is said to satisfy

- *weak l-doctrinal adjunction* if for each tight arrow $f : A \rightarrow B \in \mathbb{A}$ each adjunction $(\epsilon, Wf \dashv g, \eta)$ in \mathbb{B} lifts along W to an adjunction in \mathbb{A} with left adjoint f .
- *weak p-doctrinal adjunction* if for each tight arrow $f : A \rightarrow B \in \mathbb{A}$ each adjoint equivalence $(\epsilon, Wf \dashv g, \eta)$ in \mathbb{B} lifts along W to an adjoint equivalence in \mathbb{A} with left adjoint f .²
- *weak c-doctrinal adjunction* if for each tight arrow $f : A \rightarrow B \in \mathbb{A}$ each adjunction $(\epsilon, g \dashv Wf, \eta)$ in \mathbb{B} lifts along W to an adjunction in \mathbb{A} with right adjoint f .

²This lifting property appears biased but of course is not, since the left adjoint of an adjoint equivalence is equally its right adjoint.

The lifting properties for algebras described above can be rephrased as asserting exactly that *the forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_{p,w} \rightarrow \mathcal{C}$ satisfies weak w -doctrinal adjunction for $w \in \{l, p, c\}$* . Each of these statements asserts that if we are given a pseudomorphism of algebras whose underlying arrow has some kind of adjoint, then that adjunction lifts in a certain way; of course if the starting pseudomorphism were in fact strict then the same lifting will exist so that, in particular, *the forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ satisfies weak w -doctrinal adjunction for $w \in \{l, p, c\}$* . In fact such forgetful \mathcal{F} -functors lift these adjunctions *uniquely*. This will follow from the following simple additional lifting properties. Recall that a 2-functor $W : \mathcal{A} \rightarrow \mathcal{B}$ *reflects identity 2-cells* or is *locally conservative* when it reflects the property of a 2-cell being an identity or an isomorphism, and is *locally faithful* if it reflects the equality of parallel 2-cells. Let us say that an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ has any of these three local properties when its loose part $W_\lambda : \mathcal{A}_\lambda \rightarrow \mathcal{B}_\lambda$ has them: this means that W has these properties with respect to all 2-cells and not just those between tight morphisms. The following is evident from the definition of an algebra 2-cell.

Proposition 1. *Given $w_1, w_2 \in \{s, l, p, c\}$ satisfying $w_1 \leq w_2$ the forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_{w_1, w_2} \rightarrow \mathcal{C}$ reflects identity 2-cells, is locally conservative and locally faithful. In particular these properties are true of each $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$.*

Definition 2. Let $w \in \{l, p, c\}$. An \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ is said to satisfy *w -doctrinal adjunction* if it satisfies the unique form of weak w -doctrinal adjunction.

For instance $W : \mathbb{A} \rightarrow \mathbb{B}$ satisfies *l -doctrinal adjunction* if for each tight arrow $f : A \rightarrow B \in \mathbb{A}$ each adjunction $(\epsilon, Wf \dashv g, \eta)$ in \mathbb{B} lifts *uniquely* along W to an adjunction in \mathbb{A} with left adjoint f . Let us note that, since the $(-)^{co}$ duality interchanges left and right adjoints in an \mathcal{F} -category, W satisfies *c -doctrinal adjunction* just when W^{co} satisfies *l -doctrinal adjunction*.

Proposition 3. *Let $w \in \{l, p, c\}$ and consider $W : \mathbb{A} \rightarrow \mathbb{B}$.*

- (1) *If W satisfies weak w -doctrinal adjunction and reflects identity 2-cells it satisfies w -doctrinal adjunction.*
- (2) *If W is locally conservative and satisfies l -doctrinal adjunction or c -doctrinal adjunction then W satisfies p -doctrinal adjunction.*

Proof. (1) To prove the cases $w = l$ and $w = p$ it will suffice to show that any two adjunctions $(\epsilon, f \dashv g, \eta)$ and $(\epsilon', f \dashv g', \eta')$ in \mathbb{A} with common left adjoint and common image under W necessarily coincide in \mathbb{A} . Since adjoints are unique up to isomorphism we have $m : g \cong g'$ given by the composite

$$\begin{array}{ccc}
 & A & \xrightarrow{1} A \\
 g \nearrow & \downarrow f & \nwarrow g' \\
 B & \xrightarrow{1} B &
 \end{array}
 \quad
 \begin{array}{c}
 \epsilon \Downarrow \\
 \eta' \Downarrow
 \end{array}$$

Using the triangle equations for $f \dashv g$ and $f \dashv g'$ we see that $(mf) \cdot \eta = \eta'$ and that $\epsilon' \cdot (fm) = \epsilon$. Now the image of the 2-cell m under W is an identity by one of the triangle equations for $Wf \dashv Wg = Wg'$. Therefore m is an identity

and $g = g'$. Since mf and fm are identities it follows that $\eta = \eta'$ and $\epsilon = \epsilon'$ too. The case $w = c$ is dual.

- (2) Suppose that W satisfies l -doctrinal adjunction and is locally conservative. Then given a tight arrow $f \in \mathbb{A}$ each adjoint equivalence $(\epsilon, Wf \dashv g, \eta)$ in \mathbb{B} lifts uniquely to an adjunction $(\epsilon', f \dashv g', \eta')$ in \mathbb{A} . Since W is locally conservative both ϵ' and η' are invertible because their images are. Therefore the lifted adjunction is an adjoint equivalence so that W satisfies p -doctrinal adjunction. The c -case is dual. \square

Corollary 4. *Let $w \in \{l, p, c\}$. Then $U : \mathbf{T-Alg}_w \rightarrow \mathcal{C}$ satisfies w -doctrinal adjunction. Furthermore both $U : \mathbf{T-Alg}_l \rightarrow \mathcal{C}$ and $U : \mathbf{T-Alg}_c \rightarrow \mathcal{C}$ satisfy p -doctrinal adjunction.*

Proof. Since $U : \mathbf{T-Alg}_w \rightarrow \mathcal{C}$ satisfies weak w -doctrinal adjunction and reflects identity 2-cells it follows from Proposition 3.1 that U satisfies w -doctrinal adjunction. Since it is locally conservative the second part of the claim follows from Proposition 3.2. \square

Example 5. In the concrete setting of monoidal categories doctrinal adjunction is well known. Here we describe only those aspects relevant to our needs: namely, that the forgetful \mathcal{F} -functors $V : \mathbf{MonCat}_w \rightarrow \mathbf{Cat}$ satisfy w -doctrinal adjunction for each w . Consider then a strict monoidal functor $F : \overline{A} \rightarrow \overline{B}$ and an adjunction of categories $(\epsilon, F \dashv G, \eta)$. The right adjoint G obtains the structure of a lax monoidal functor $\overline{G} = (G, g, g_0) : \overline{B} \rightsquigarrow \overline{A}$ with constraints $g_{x,y}$ and g_0 given by the composites

$$GxGy \xrightarrow{\eta_{GxGy}} GF(GxGy) = G(FGxFGy) \xrightarrow{G(\epsilon_x \epsilon_y)} G(xy) \quad i^A \xrightarrow{\eta_i^A} GF i^A = G i^B$$

It is straightforward to show that, with respect to these constraints, the natural transformations ϵ and η become monoidal transformations. Therefore we obtain the lifted adjunction $(\epsilon, F \dashv (G, g, g_0), \eta)$ in \mathbf{MonCat}_l whose uniqueness follows, using Proposition 3.1, from the fact that V reflects identity 2-cells; thus $V : \mathbf{MonCat}_l \rightarrow \mathbf{Cat}$ satisfies l -doctrinal adjunction. The cases $w = p$ and $w = c$ are entirely analogous.

Note that unless $w = l$ the forgetful \mathcal{F} -functor $V : \mathbf{MonCat}_w \rightarrow \mathbf{Cat}$ does not satisfy l -doctrinal adjunction. That $V : \mathbf{MonCat}_l \rightarrow \mathbf{Cat}$ itself does so is due to the fact that the constraints $f_{a,b} : FaFb \rightarrow F(ab)$ and $f_0 : i^B \rightarrow Fi^A$ point in the correct direction – into F – and are not invertible. While l -doctrinal adjunction captures, to some extent, this *laxness* it does not determine in any way the *coherence axioms* for a lax monoidal functor. This is illustrated by the fact that there is an \mathcal{F} -category of monoidal categories, strict and *incoherent* lax monoidal functors (these have components f and f_0 oriented as above but satisfying no equations) and this too sits over \mathbf{Cat} via a forgetful \mathcal{F} -functor satisfying l -doctrinal adjunction.

3.2. Morphisms of adjunctions and w -doctrinal \mathcal{F} -functors. Although each forgetful \mathcal{F} -functor $U : \mathbf{T-Alg}_w \rightarrow \mathcal{C}$ satisfies w -doctrinal adjunction it turns out

that, as far as this property characterises such \mathcal{F} -functors, the only relevant adjunctions have identity units or counits. To capture this we say that an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ satisfies

- *l-Sect* if given a tight morphism $f : A \rightarrow B \in \mathbb{A}$ each adjunction of the form $(1, Wf \dashv g, \eta) \in \mathbb{B}$ lifts uniquely to an adjunction $(1, f \dashv g', \eta')$ of the same form in \mathbb{A} .
- *p-Sect* if given a tight morphism $f : A \rightarrow B \in \mathbb{A}$ each adjoint equivalence of the form $(1, Wf \dashv g, \eta) \in \mathbb{B}$ lifts uniquely to an adjoint equivalence $(1, f \dashv g', \eta')$ of the same form in \mathbb{A} .
- *c-Sect* if given a tight morphism $g : A \rightarrow B \in \mathbb{A}$ each adjunction of the form $(\epsilon, f \dashv Wg, 1) \in \mathbb{B}$ lifts uniquely to an adjunction $(\epsilon', f' \dashv g, 1)$ of the same form in \mathbb{A} .

Whilst we only need concern ourselves with liftings of the above kinds of adjoint sections we will also need, in Theorem 17, to consider liftings of *morphisms of adjunctions*. Given adjunctions $(\epsilon_1, f_1 \dashv g_1, \eta_1)$ and $(\epsilon_2, f_2 \dashv g_2, \eta_2)$ in a 2-category we call a commutative square $(r, s) : f_1 \rightarrow f_2$

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ f_1 \downarrow & & \downarrow f_2 \\ C & \xrightarrow{s} & D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{s} & D \\ g_1 \downarrow & & \downarrow g_2 \\ A & \xrightarrow{r} & B \end{array}$$

a *morphism of left adjoints* from f_1 to f_2 and a commuting square $(s, r) : g_1 \rightarrow g_2$ a *morphism of right adjoints*. If the pair (r, s) satisfies both of these conditions as well as the compatibilities $s\epsilon_1 = \epsilon_2 s$ and $r\eta_1 = \eta_2 r$ with the units and counits, then (r, s) constitutes a *morphism of adjunctions* $(r, s) : (\epsilon_1, f_1 \dashv g_1, \eta_1) \rightarrow (\epsilon_2, f_2 \dashv g_2, \eta_2)$.³ The following lemma is sometimes useful for recognising morphisms of adjunctions.

Lemma 6. *Let $r : A \rightarrow B$ and $s : C \rightarrow D$. Then (r, s) constitutes a morphism of adjunctions $(r, s) : (\epsilon_1, f_1 \dashv g_1, \eta_1) \rightarrow (\epsilon_2, f_2 \dashv g_2, \eta_2)$ just when either*

- (1) *$(r, s) : f_1 \rightarrow f_2$ is a morphism of left adjoints with mate an identity.*
- (2) *$(s, r) : g_1 \rightarrow g_2$ is a morphism of right adjoints with mate an identity.*

Proof. It suffices to prove (1) – given a morphism of adjunctions $(r, s) : f_1 \rightarrow f_2$

$$\begin{array}{ccccc} \begin{array}{ccc} A & \xrightarrow{r} & B \\ f_1 \downarrow & & \downarrow f_2 \\ C & \xrightarrow{s} & D \end{array} & \begin{array}{ccccc} & A & \xrightarrow{r} & B & \xrightarrow{1} & B \\ & g_1 \nearrow & \downarrow f_1 & \downarrow f_2 & \searrow \eta_2 \Downarrow & \\ C & \xrightarrow{1} & C & \xrightarrow{s} & D \end{array} & \begin{array}{ccccc} & A & \xrightarrow{1} & A & \xrightarrow{r} & B \\ & g_1 \nearrow & \downarrow f_1 & \downarrow f_1 & \searrow \eta_1 \Downarrow & \\ C & \xrightarrow{1} & C & \xrightarrow{s} & D \end{array} & \begin{array}{ccc} A & \xrightarrow{r} & B \\ & \nearrow g_2 & \\ C & \xrightarrow{s} & D \end{array} \end{array}$$

consider the mate of the left square: the central composite above. As (r, s) is a morphism of adjunctions we have $\eta_2 r = r \eta_1$ so that the mate reduces to the rightmost composite: this is an identity by the triangle equation for $f_1 \dashv g_1$.

Conversely if the mate of $(r, s) : f_1 \rightarrow f_2$ is an identity then it exhibits $(s, r) : g_1 \rightarrow g_2$ as a morphism of right adjoints. It then remains to verify the unit and counit compatibilities for a morphism of adjunctions: this is straightforward. \square

³These are often called *strict* morphisms of adjunctions but, as they are the only kind we consider, we call them morphisms of adjunctions.

Now adjunctions $(\epsilon_1, f_1 \dashv g_1, \eta_1)$ and $(\epsilon_2, f_2 \dashv g_2, \eta_2)$ in an \mathcal{F} -category \mathbb{A} are just adjunctions in \mathcal{A}_λ – likewise a pair (r, s) constitute a morphism of left/right adjoints or adjunctions in \mathbb{A} just when this is true in \mathcal{A}_λ . If, however, both r and s are tight we call (r, s) a *tight morphism* of left/right adjoints or adjunctions in \mathbb{A} . Like 2-functors each \mathcal{F} -functor preserves morphisms of adjoints and adjunctions but also tight morphisms of adjoints and adjunctions. We say that an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ satisfies

- *l-Morph* if given adjunctions $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in \mathbb{A} with tight left adjoints, a tight morphism $(r, s) : f_1 \rightarrow f_2$ of left adjoints is a morphism of adjunctions just when $(Wr, Ws) : Wf_1 \rightarrow Wf_2$ is one in \mathbb{B} .
- *p-Morph* if given adjoint equivalences $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in \mathbb{A} with tight left adjoints, a tight morphism $(r, s) : f_1 \rightarrow f_2$ of left adjoints is a morphism of adjunctions just when $(Wr, Ws) : Wf_1 \rightarrow Wf_2$ is one in \mathbb{B} .
- *c-Morph* if given adjunctions $(\epsilon_1, f_1 \dashv g_1, 1)$ and $(\epsilon_2, f_2 \dashv g_2, 1)$ in \mathbb{A} with tight right adjoints, a tight morphism $(r, s) : g_1 \rightarrow g_2$ of right adjoints is a morphism of adjunctions just when $(Wr, Ws) : Wg_1 \rightarrow Wg_2$ is one in \mathbb{B} .

Definition 7. Let $w \in \{l, p, c\}$. An \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathbb{B}$ is said to be *w-doctrinal* if it satisfies *w-Sect*, *w-Morph* and is locally faithful. We denote the class of *w-doctrinal* \mathcal{F} -functors by *w-Doct*.

The condition that W be locally faithful may seem somewhat unnatural – see the discussion after Theorem 17 for our reasons for including it.

Note that W is *c-doctrinal* just when W^{co} is *l-doctrinal*. Let us now compare these lifting properties with those of Section 3.1.

Lemma 8. Let $w \in \{l, p, c\}$ and consider $W : \mathbb{A} \rightarrow \mathbb{B}$.

- (1) If W satisfies *w-doctrinal adjunction* and reflects identity 2-cells then it satisfies *w-Sect* and *w-Morph*.
- (2) If W satisfies *w-doctrinal adjunction*, reflects identity 2-cells and is locally faithful then it is *w-doctrinal*.
- (3) If W is locally conservative, reflects identity 2-cells, is locally faithful and satisfies either *l* or *c-doctrinal adjunction* then it is *p-doctrinal*.

Proof. We will only consider the *l*-case of (1) and (2), all being essentially identical, with the *l* and *c* cases dual.

- (1) Given a tight morphism $f : A \rightarrow B \in \mathbb{A}$ and an adjunction $(1, Wf \dashv g, \eta)$ we know, as W satisfies *l-doctrinal adjunction*, that the adjunction lifts uniquely to an adjunction $(\epsilon', f \dashv g', \eta')$. Since $W\epsilon' = 1$ and W reflects identity 2-cells ϵ' is an identity. This verifies *l-Sect*. For *l-Morph* consider adjunctions $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in \mathbb{A} with tight left adjoints and a tight morphism $(r, s) : f_1 \rightarrow f_2$ of left adjoints such that $(Wr, Ws) : Wf_1 \rightarrow Wf_2$ is a morphism of adjunctions; we must show $(r, s) : f_1 \rightarrow f_2$ is too. By Lemma 6 this is equally to say that the mate $m_{r,s} : rg_1 \Rightarrow g_2s$ of this square is an identity 2-cell, so giving a commutative square $(s, r) : g_1 \rightarrow g_2$. So it will suffice to check $Wm_{r,s}$ is an identity. But $Wm_{r,s} = m_{Wr, Ws}$ which is an identity since (Wr, Ws) is a morphism of adjunctions.
- (2) Since *w-doctrinal* just means *w-Sect* and *w-Morph* together with local faithfulness this follows immediately from Part 1.

- (3) Since by Proposition 3.2 such an \mathcal{F} -functor satisfies p -doctrinal adjunction this follows from Part 2. \square

Corollary 9. *Let T be a 2-monad on \mathcal{C} . For each $w \in \{l, p, c\}$ the forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ is w -doctrinal. Furthermore both $U : \mathbf{T}\text{-Alg}_l \rightarrow \mathcal{C}$ and $U : \mathbf{T}\text{-Alg}_c \rightarrow \mathcal{C}$ are p -doctrinal.*

Proof. Since $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ satisfies w -doctrinal adjunction, reflects identity 2-cells and is locally conservative the first claim follows from Lemma 8.2; since each such U is also locally conservative the second claim follows from Lemma 8.3. \square

Let us conclude by mentioning a further evident example of w -doctrinal \mathcal{F} -functors.

Proposition 10. *Let $w \in \{l, p, c\}$. If $W : \mathbb{A} \rightarrow \mathbb{B}$ is such that $W_\lambda : \mathcal{A}_\lambda \rightarrow \mathcal{B}_\lambda$ is 2-fully faithful then W is w -doctrinal for each w . In particular each equivalence of \mathcal{F} -categories is w -doctrinal for each w .*

3.3. A small orthogonality class. Though not strictly necessary in what follows let us remark that, with the exception of weak w -doctrinal adjunction, all of the lifting/reflection properties so far considered are expressible as orthogonal lifting properties in $\mathcal{F}\text{-CAT}$. Certainly it is not hard to see that this is true of the property of reflecting identity 2-cells or of being locally faithful or of being locally conservative. Less obvious is that this is true of the notion of w -doctrinal adjunction or of the conditions $w\text{-Sect}$ and $w\text{-Morph}$, in particular of the condition $w\text{-Morph}$ concerning liftings of morphisms of adjunctions. We describe the conditions $l\text{-Sect}$ and $l\text{-Morph}$ here – the p and c cases being similar. To this end consider the following \mathcal{F} -category $\mathbb{A}dj_l$ depicted in its entirety on the left below

$$\begin{array}{ccc}
 \begin{array}{ccc} 0 & \xrightarrow{f} & 1 \\ \downarrow 1 & \overset{\eta}{\Rightarrow} & \downarrow 1 \\ 0 & \xleftarrow{g} & 1 \end{array} & \begin{array}{ccc} \mathbf{2} & \xrightarrow{j} & \mathbb{A}dj_l \\ \downarrow & \swarrow & \downarrow \\ \mathbb{A} & \xrightarrow{W} & \mathbb{B} \end{array} & \begin{array}{ccc} \mathcal{F}\text{-CAT}(\mathbb{A}dj_l, \mathbb{A}) & \xrightarrow{W_*} & \mathcal{F}\text{-CAT}(\mathbb{A}dj_l, \mathbb{B}) \\ j^* \downarrow & & \downarrow j^* \\ \mathcal{F}\text{-CAT}(\mathbf{2}, \mathbb{A}) & \xrightarrow{W_*} & \mathcal{F}\text{-CAT}(\mathbf{2}, \mathbb{B}) \end{array}
 \end{array}$$

where gf is loose, $fg = 1$, $f\eta = 1$ and $\eta g = 1$. This is the free adjunction with identity counit and tight left adjoint f . It has a single non-identity tight arrow f so that $\mathbb{A}dj_{l\tau}$ equals the free tight arrow $\mathbf{2}$; therefore the inclusion $\mathbb{A}dj_{l\tau} \rightarrow \mathbb{A}dj_l$ is the \mathcal{F} -functor $j : \mathbf{2} \rightarrow \mathbb{A}dj_l$ selecting f . Now to give a commutative square as in the middle above is to give a tight arrow $f \in \mathbb{A}$ and adjunction $(1, Wf \dashv g, \eta)$ in \mathbb{B} . To give a filler is to lift the adjunction to an adjunction $(1, f \dashv g', \eta')$; thus W is orthogonal to j when condition $l\text{-Sect}$ is met. The conditions $l\text{-Sect}$ and $l\text{-Morph}$ together assert exactly that j and W are orthogonal in $\mathcal{F}\text{-CAT}$ as a 2-category – this means that the right square above is a pullback in CAT . In fact by a standard argument the conditions $l\text{-Sect}$ and $l\text{-Morph}$ jointly amount to ordinary (1-categorical) orthogonality against the single \mathcal{F} -functor $j \times 1 : \mathbf{2} \times \mathbf{2} \rightarrow \mathbb{A}dj_l \times \mathbf{2}$, of which j is a retract.

Therefore the class of w -doctrinal \mathcal{F} -functors, $w\text{-Doct}$, forms a right orthogonality class in the category of \mathcal{F} -categories and \mathcal{F} -functors. The following section is geared towards establishing sufficient conditions on an \mathcal{F} -category \mathbb{A} under which

the inclusion $j : \mathcal{A}_\tau \rightarrow \mathbb{A}$ belongs to ${}^\perp w\text{-Doct}$ – is orthogonal to each w -doctrinal \mathcal{F} -functor.

4. REPRESENTING LOOSE MORPHISMS BY TIGHT SPANS

We now consider completeness properties of \mathcal{F} -categories appropriate to those of the form $\mathbf{T}\text{-Alg}_w$. In an \mathcal{F} -category \mathbb{A} with such completeness properties we can represent loose morphisms in \mathbb{A} by tight spans. In the lax setting, for instance, each loose morphism $f : A \rightsquigarrow B$ is represented as a tight span $C_f : A \rightarrow B$

$$\begin{array}{ccc} & C_f & \\ p_f \swarrow & & \searrow q_f \\ A & & B \end{array}$$

by taking its *colax* limit. The tight left leg p_f then has a loose right adjoint $r_f : A \rightsquigarrow C_f$ from which f can be recovered as the composite $q_f r_f : A \rightsquigarrow C_f \rightarrow B$. This ability to so represent and recover loose morphisms will play a crucial role in our key result, Theorem 17.

4.1. \mathcal{F} -categorical limits. In this section and the next we will make use of a few \mathcal{F} -categorical limits – w -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$ – so let us begin with a small overview of limits in \mathcal{F} -categories.

\mathcal{F} -categories were introduced in [15] because the behaviour of limits in the 2-category $\mathbf{T}\text{-Alg}_w$ is \mathcal{F} -categorical rather than 2-categorical. If, for example, the base \mathcal{C} admits pseudolimits of arrows (see Section 4.2 for details) then given a pseudomorphism $f : A \rightsquigarrow B \in \mathbf{T}\text{-Alg}_p$ we can form its 2-categorical pseudolimit P_f in $\mathbf{T}\text{-Alg}_p$

$$\begin{array}{ccc} & P_f & \\ p_f \swarrow & \cong_{\lambda_f} & \searrow q_f \\ A & \rightsquigarrow_f & B \end{array}$$

by lifting it from the base. When we do so it turns out that the *limit projections* p_f and q_f are *strict algebra morphisms* and *jointly detect strictness*; indeed the property of *limit projections being tight and jointly detecting tightness* is exactly that which distinguishes \mathcal{F} -categorical limits (in the sense of \mathcal{F} -enriched category theory) from 2-categorical ones (see Proposition 3.6 of [15]). The manner in which the pseudolimit of f was formed over the base can then be understood as asserting exactly that $U : \mathbf{T}\text{-Alg}_p \rightarrow \mathcal{C}$ creates *pseudolimits of loose morphisms*. Theorem 5.13 of [15] gives a complete characterisation of those limits created by the forgetful \mathcal{F} -functors $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$. We will only need to know that the following few limits lift, and that these lift goes back further to [2] and [12].

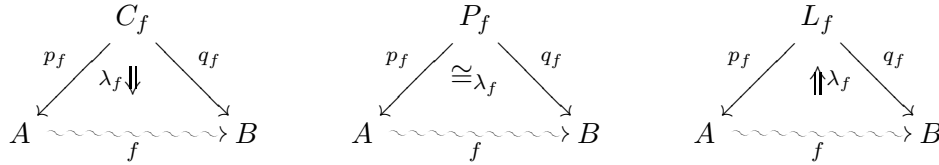
Proposition 11. *Let $w \in \{l, p, c\}$. The forgetful \mathcal{F} -functor $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ creates all tight limits, which include tight pullbacks and cotensors with $\mathbf{2}$, and also creates \bar{w} -limits of loose morphisms.*

These terms will be defined precisely in this and the next section. We draw the reader's attention to two points here. *Tight limits*, as in the pullback of a opspan of tight morphisms but not loose ones, are very common: by the above proposition

they exist in all \mathcal{F} -categories of algebras when they exist in the base. On the other hand \overline{w} -limits of loose morphisms – the \overline{p} -limit of f is its pseudolimit as above – are less widespread, and indeed their behaviour distinguishes the different \mathcal{F} -categories of algebras.

4.2. w-limits of loose morphisms. There are three limits to consider here – *colax*, *pseudo* and *lax limits of loose morphisms* – these correspond, in turn, to lax, pseudo and colax morphisms. As we always lead with the case of lax morphisms we focus here primarily on colax limits.

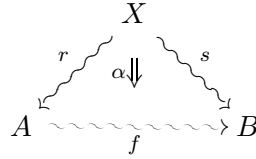
Given a loose morphism $f : A \rightsquigarrow B \in \mathbb{A}$ its (colax/pseudo/lax)-limit consists of an object $C_f/P_f/L_f$ and a cone (p_f, λ_f, q_f) as below



with both projections p_f and q_f *tight*.

The colax limit C_f ⁴ is required to be the usual 2-categorical colax limit [12] in \mathcal{A}_λ : this means that it has the following two universal properties.

- (1) Given any cone (r, α, s)



there exists a unique $t : X \rightsquigarrow C_f$ satisfying

$$p_f t = r, q_f t = s \text{ and } \lambda_f t = \alpha$$

- (2) Given a pair of cones (r, α, s) and (r', α', s') with common base X together with 2-cells $\theta_r : r \Rightarrow r' \in \mathcal{A}_\lambda(X, A)$ and $\theta_s : s \Rightarrow s' \in \mathcal{A}_\lambda(X, B)$ satisfying

$$\begin{array}{ccc} s & \xRightarrow{\theta_s} & s' \\ \alpha \Downarrow & & \Downarrow \alpha' \\ fr & \xRightarrow{f\theta_r} & fr' \end{array}$$

there exists a unique 2-cell $\phi : t \Rightarrow t' \in \mathcal{A}_\lambda(X, C_f)$ between the induced factorisations such that

$$p_f \phi = \theta_r \text{ and } q_f \phi = \theta_s .$$

For C_f to be the colax limit of f in the \mathcal{F} -categorical sense we must also have

- (3) A morphism $t : X \rightsquigarrow C_f$ is tight just when $p_f t$ and $q_f t$ are tight – *the projections jointly detect tightness*.

⁴Colax limits of arrows are usually called oplax limits of arrows. We prefer colax limits here since they are lax limits in \mathbb{A}^{co} rather than \mathbb{A}^{op} and, similarly, sit better with our usage of lax and colax morphisms.

In the case of the pseudolimit P_f the 2-cell λ_f is required to be invertible. If we call those cones with an invertible 2-cell *pseudo-cones* then the universal properties of (1) and (2) above are only changed by replacing cones by pseudo-cones – thus (p_f, λ_f, q_f) is the universal pseudo-cone. The \mathcal{F} -categorical aspect of (3) remains the same.

The lax limit of f is the colax limit in \mathbb{A}^{co} . As this case can be treated entirely by duality we will avoid it from now on. Omitting lax limits means that we can cover the limiting cones (p_f, λ_f, q_f) for colax and pseudo limits by the single diagram shape

$$\begin{array}{ccc} & W_f & \\ p_f \swarrow & \lambda_f \Downarrow & \searrow q_f \\ A & \text{---} f \text{---} & B \end{array}$$

where the reader should interpret λ_f as invertible in case of the pseudolimit – $w = p$.

We mentioned above that lax morphisms correspond to colax limits and vice-versa. For $w \in \{l, p, c\}$ let us set $\bar{l} = c$, $\bar{p} = p$ and $\bar{c} = l$ as in [15]: this correspondence is then captured, as in Proposition 11, by the fact that for each $w \in \{l, p, c\}$ the forgetful \mathcal{F} -functor $U : \mathbf{T-Alg}_w \rightarrow \mathcal{C}$ creates \bar{w} -limits of loose morphisms.

Example 12. In \mathbf{Cat} the colax limit of a functor $F : A \rightarrow B$ is given by the comma category B/F : this has objects $(\alpha : x \rightarrow Fa, a)$ and morphisms $(r, s) : (x, \alpha, a) \rightarrow (y, \beta, b)$ given by pairs of arrows $r : x \rightarrow y \in \mathcal{B}$ and $s : a \rightarrow b \in \mathcal{A}$ rendering commutative the square

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & Fa \\ r \downarrow & & \downarrow Fs \\ y & \xrightarrow{\beta} & Fb \end{array} \qquad \begin{array}{ccc} & B/F & \\ p \swarrow & \lambda \Downarrow & \searrow q \\ A & \xrightarrow{F} & B \end{array}$$

The projections $p : B/F \rightarrow A$ and $q : B/F \rightarrow B$ of the limit cone (p, λ, q) act on a morphism (r, s) of B/F as $p(r, s) = s : a \rightarrow b$ and $q(r, s) = r : x \rightarrow y$; the value of $\lambda : q \Rightarrow pF$ at (a, α, b) is simply the morphism $\alpha : x \rightarrow Fa$ itself.

The pseudolimit of F is the full subcategory of B/F whose objects are those pairs $(\alpha : x \rightarrow Fa, a)$ with α invertible, whilst the lax limit of F is the comma category F/B .

Example 13. It is illuminating to consider the colax limit of a lax monoidal functor $F = (F, f, f_0) : \bar{A} \rightsquigarrow \bar{B}$. The forgetful \mathcal{F} -functor $U : \mathbf{MonCat}_l \rightarrow \mathbf{Cat}$ creates these limits: to see how this goes first consider the colax limit of the functor F , the comma category B/F equipped with its limiting cone (p, λ, q) described above. The crux of the argument is to show that this lifts uniquely to a cone in \mathbf{MonCat}_l : that B/F admits a unique monoidal structure such that p and q become strict monoidal and λ a monoidal transformation.

So consider two objects (x, α, a) and (y, β, b) of B/F : if p and q are to be strict monoidal the tensor product $(x, \alpha, a)(y, \beta, b)$ must certainly be of the form (xy, θ, ab) ; furthermore the tensor condition for λ to be a monoidal transformation interpreted

at this pair asserts precisely that $(x, \alpha, a)(y, \beta, b)$ equals

$$xy \xrightarrow{\alpha\beta} FaFb \xrightarrow{f_{a,b}} F(ab)$$

Likewise the unit condition for a monoidal transformation forces us to define the unit of B/F to be $(f_0 : i^B \rightarrow Fi^A, i^A)$. For p and q to preserve tensor products of morphisms we must define the tensor product as $(r, s)(r', s') = (rr', ss')$ at morphisms of B/F – to say the resulting pair is a morphism of B/F is then to say that the following square is commutative.

$$\begin{array}{ccccc} xy & \xrightarrow{\alpha\beta} & FaFb & \xrightarrow{f_{a,b}} & F(ab) \\ rr' \downarrow & & \downarrow FsFs' & & \downarrow F(ss') \\ x'y' & \xrightarrow{\alpha'\beta'} & Fa'Fb' & \xrightarrow{f_{a',b'}} & F(a'b') \end{array}$$

The left square trivially commutes and the right square commutes by naturality of the $f_{a,b}$. It remains to give the associator and the left and right unit constraints for the monoidal structure on B/F – this is where the coherence axioms for a lax monoidal functor finally come into play. Certainly if p and q are to be strict monoidal they must preserve the associators strictly: this means that the associator at a triple of objects $((x, \alpha, a), (y, \beta, b), (z, \gamma, c))$ of B/F must be given by $(\lambda_{x,y,z}^B, \lambda_{a,b,c}^A)$. To say that this is a morphism of B/F is equally to say that the composite square

$$\begin{array}{ccccccc} (xy)z & \xrightarrow{(\alpha\beta)\gamma} & (FaFb)Fc & \xrightarrow{f_{a,b}1} & F(ab)Fc & \xrightarrow{f_{ab,c}} & F((ab)c) \\ \lambda_{x,y,z}^B \downarrow & & \downarrow \lambda_{Fa,Fb,Fc}^B & & & & \downarrow F\lambda_{a,b,c}^A \\ x(yz) & \xrightarrow{\alpha(\beta\gamma)} & Fa(FbFc) & \xrightarrow{1f_{b,c}} & FaF(bc) & \xrightarrow{f_{a,bc}} & F(a(bc)) \end{array}$$

is commutative. The left square commutes by naturality of the associators in B with the right square asserting exactly the associativity condition for a lax monoidal functor. Similarly the left and right unit constraints at (x, α, a) must be given by $(\rho_l^B x, \rho_l^A a)$ and $(\rho_r^B x, \rho_r^A a)$ – that these lift to isomorphisms of B/F likewise correspond to the left and right unit conditions for a lax monoidal functor. Having given the monoidal structure for B/F it remains to check it verifies the axioms for a monoidal category, but all of these clearly follow from the corresponding axioms for \overline{A} and \overline{B} because p and q preserve the structure strictly and are jointly faithful.

Finally one needs to verify that this uniquely lifted cone in MonCat_l satisfies the universal property of the colax limit of (F, f, f_0) therein. That p and q jointly detect tightness follows from the fact that they jointly reflect identity arrows – from here it is straightforward to verify that B/F has the universal property of the colax limit in MonCat_l . For $w \in \{p, c\}$ one constructs the \overline{w} -limit of a loose morphism in MonCat_w in an entirely similar way.

Observe that in lifting the above cone (p, λ, q) to MonCat_l we used all of the coherence axioms for a lax monoidal functor, and indeed these generating coherence axioms are required for the cone to lift. Thus while l -doctrinal adjunction is related to the *laxness* – orientation and non-invertibility – of our lax monoidal functors,

colax limits of loose morphisms concern the *coherence* axioms these lax morphisms must satisfy.

4.3. Recovering loose morphisms from tight spans. Let $w \in \{p, c\}$ and suppose that \mathbb{A} admits w -limits of loose morphisms. Then given $f : A \rightsquigarrow B$ in \mathbb{A} we have the commutative triangle on the left below

By the universal property of W_f we obtain a unique 1-cell $r_f : A \rightsquigarrow W_f$ satisfying

$$p_f r_f = 1, \quad q_f r_f = f \text{ and } \lambda_f r_f = 1 \quad (4.1)$$

as expressed in the equality of pasting diagrams above. Since p_f and q_f jointly detect tightness we also have that

$$r_f \text{ is tight just when } f \text{ is.} \quad (4.2)$$

Proposition 14. *Consider $f : A \rightsquigarrow B$ as above.*

- (1) *In the case of the colax limit C_f we have an adjunction $(1, p_f \dashv r_f, \eta_f)$ where $\eta_f : 1 \Rightarrow r_f p_f$ is the unique 2-cell satisfying*

$$p_f \eta_f = 1 \text{ and } q_f \eta_f = \lambda_f. \quad (4.3)$$

- (2) *In the case of the pseudolimit P_f the identical equations hold with the exception that η_f is invertible; thus $(1, p_f \dashv r_f, \eta_f)$ is an adjoint equivalence.*

Proof.

- (1) We need to give a unit $\eta_f : 1 \Rightarrow r_f p_f$. To give such a 2-cell is, by the 2-dimensional universal property of C_f , equally to give 2-cells $\theta_1 : p_f(1) \Rightarrow p_f(r_f p_f)$ and $\theta_2 : q_f(1) \Rightarrow q_f(r_f p_f)$ satisfying $\theta_1 \cdot \lambda_f = (\lambda_f r_f p_f) \cdot \theta_2$. We take θ_1 to be the identity and θ_2 to be $\lambda_f : q_f \Rightarrow f p_f$; the required equality involving θ_1 and θ_2 is then the assertion that λ_f equals itself. We thus obtain a unique $\eta_f : 1 \Rightarrow r_f p_f$ such that $p_f \eta_f = 1$ and $q_f \eta_f = \lambda_f$. If the identity 2-cell $p_f q_f = 1$ is to be the counit of the adjunction then the triangle equations become $p_f \eta_f = 1$ and $\eta_f r_f = 1$. So it remains to check that $\eta_f r_f = 1$ for which it suffices, again by the 2-dimensional universal property of C_f , to show that $p_f \eta_f r_f = 1$ and $q_f \eta_f r_f = 1$. The first of these holds since $p_f \eta_f = 1$; the second since $q_f \eta_f = \lambda_f$ and $\lambda_f r_f = 1$.
- (2) This case is essentially identical – the key point is that the 2-cells $\theta_1 = 1$ and $\theta_2 = \lambda_f$ used above to construct η_f are now both invertible. That η_f is itself invertible follows from the fact that p_f and q_f are jointly conservative – this conservativity follows from the 2-dimensional universal property of P_f . \square

Let us remark that if we ignore \mathcal{F} -categorical aspects then the above constructions and resulting factorisations $f = q_f r_f$ have appeared in various contexts. In the pseudolimit case the factorisation is the (trivial cofibration, fibration)-factorisation of the natural model structure on a 2-category [13]. In \mathbf{Cat} the factorisation $q_f r_f : A \rightarrow C_f \rightarrow B$ of a functor f through its colax limit coincides with its factorisation $A \rightarrow B/f \rightarrow B$ through the comma category B/f – this is the factorisation (Lf, Rf) of a natural weak factorisation system on \mathbf{Cat} described in [4].

4.4. Embedding into tight spans. We now turn to the functoriality of the assignment which sends a loose morphism $f : A \rightsquigarrow B$ to the tight span $W_f : A \rightrightarrows B$. To consider composition of these tight spans we will need tight pullbacks. Given tight morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$ in \mathbb{A} the *tight pullback* D of f and g

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ q \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

is the pullback in the 2-category \mathcal{A}_λ with, moreover, both projections p and q tight and jointly detecting tightness. This is equally to say that D is a pullback in \mathcal{A}_τ which is preserved by the inclusion $j : \mathcal{A}_\tau \rightarrow \mathcal{A}_\lambda$. Being tight limits, tight pullbacks are created by each forgetful \mathcal{F} -functor $U : \mathbf{T-Alg}_w \rightarrow \mathcal{C}$ (see Proposition 11).

In particular when \mathbb{A} admits tight pullbacks we can consider the bicategory of *tight spans*, $\mathbf{Span}(U\mathcal{A}_\tau)$, in the underlying category $U\mathcal{A}_\tau$ of \mathcal{A}_τ . We then have

Theorem 15. *Let $w \in \{p, c\}$ and suppose that \mathbb{A} has w -limits of loose morphisms and tight pullbacks. The map sending $f : A \rightsquigarrow B$ to the tight span $W_f : A \rightrightarrows B$ is the action on 1-cells of an identity on objects lax functor $W : U\mathcal{A}_\lambda \rightarrow \mathbf{Span}(U\mathcal{A}_\tau)$.*

Proof. Given $f : A \rightsquigarrow B$ we have our tight span $W_f : A \rightrightarrows B$

$$\begin{array}{ccc} & W_f & \\ p_f \swarrow & & \searrow q_f \\ A & & B \end{array}$$

Given a composable pair $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ we need our lax functor constraint, a span map $k_{g,f} : W_g W_f \rightarrow W_{gf}$ where $W_g W_f$ is the span composite of W_g and W_f given by the pullback centre left below

$$\begin{array}{ccc} & W_g W_f & \\ p_{g,f} \swarrow & & \searrow q_{g,f} \\ W_f & & W_g \\ p_f \swarrow \lambda_f \Downarrow \searrow q_f & & p_g \swarrow \lambda_g \Downarrow \searrow q_g \\ A \rightsquigarrow f \rightsquigarrow B & & B \rightsquigarrow g \rightsquigarrow C \end{array} = \begin{array}{ccc} & W_g W_f & \\ p_{g,f} \swarrow & \downarrow k_{g,f} & \searrow q_{g,f} \\ W_f & W_{gf} & W_g \\ p_f \swarrow p_{gf} \searrow \lambda_{gf} \Downarrow & & q_{gf} \searrow q_g \\ A \rightsquigarrow f \rightsquigarrow B & & B \rightsquigarrow g \rightsquigarrow C \end{array}$$

By the universal property of the colax limit W_{gf} the composite 2-cell of the left pasting diagram induces a unique *tight* morphism $k_{g,f} : W_g W_f \rightarrow W_{gf}$ satisfying

the equations

$$p_{gf}k_{g,f} = p_f p_{g,f}, q_{gf}k_{g,f} = q_g q_{g,f} \text{ and } \lambda_{gf}k_{g,f} = (g\lambda_f p_{g,f}).(\lambda_g q_{g,f}) \quad (4.4)$$

equally expressed in the equality of pasting diagrams. This constructs all of the lax functor that we will need – we leave the remaining details to the reader. \square

It is worth noting that in the case of colax limits the lax functor $C : UA_\lambda \rightarrow Span(UA_\tau)$ extends naturally to 2-cells giving a lax functor $C : \mathcal{A}_\lambda \rightarrow Span(UA_\tau)$: at $\alpha : f \Rightarrow g \in \mathcal{A}_\lambda$ the composite $\alpha p_f \cdot \lambda_f : q_f \Rightarrow g p_f$ induces, by the universal property of C_g , the required span map $C_\alpha : C_f \rightarrow C_g$. However the corresponding construction only works in the pseudo-case if α is invertible: this is because P_g only has its universal property with respect to pseudo-cones. To account for this failure in the pseudo-case we will employ cotensors with $\mathbf{2}$ in the following section. Again let us specialise from \mathcal{F} -categories \mathbb{A} to 2-categories \mathcal{A} to compare these constructions with better known ones. Then we are looking at 2-categories with w -limits of arrows and pullbacks. In the case $w = l$ or $w = c$ we find that both completeness criteria coincide in agreeing with Gray's notion of a representable 2-category [5]. Gray discussed an embedding, further treated by Street in [19], of a representable 2-category \mathcal{A} into the 2-category of categories $Cat(UA)$ internal to \mathcal{A} . As internal categories relate to spans his embedding $\mathcal{A} \rightarrow Cat(UA)$ naturally compares to our embedding $\mathcal{A} \rightarrow Span(UA)$. We note that there is a natural variant of Gray's embedding which attempts to embed \mathcal{A} into internal groupoids therein – here one encounters the same problems with representing arbitrary 2-cells via internal natural transformations that occurred in the pseudo-setting above.

4.5. Tight pullbacks versus w -limits of composable pairs of loose morphisms. Let us briefly remark that for our applications we do not require our \mathcal{F} -categories to admit *all* tight pullbacks but only enough to obtain the span composite $W_g W_f$ of W_g and W_f . This is equally the *w-limit of the composable pair* $(f, g) : A \rightsquigarrow B \rightsquigarrow C$: when $w = c$, for instance, this is the universal diagram of shape

$$\begin{array}{ccccc} & & C_g C_f & & \\ & \swarrow & \downarrow & \searrow & \\ A & \rightsquigarrow_f & B & \rightsquigarrow_g & C \end{array}$$

Whenever the colax limits C_f and C_g exist, the above 2-cells respectively induce tight morphisms $p_{g,f} : C_g C_f \rightarrow C_f$ and $q_{g,f} : C_g C_f \rightarrow C_g$ and, comparing universal properties, one sees that these exhibit $C_g C_f$ as the tight pullback of q_f and p_g . Since these tight pullbacks are the only ones we require to exist in our applications, all based upon Theorem 17, we can replace *w-limits of loose morphisms and tight pullbacks* wherever they appear in the following theorems – notably in the monadicity theorems of Section 6 – by *w-limits of loose morphisms and w-limits of composable pairs of loose morphisms* without altering the validity of those results. Whether these slightly weaker completeness assumptions will usefully extend the applicability of our results is not clear so we have preferred to work with tight pullbacks.

4.6. All morphisms of left adjoints are morphisms of adjunctions. Our calculations to this point will enable us to construct the diagonal filler of Theorem 17 on 1-cells, but are not enough to establish its functoriality. For this we also need to know that the other commuting squares of left adjoints, as arose in the composition of tight spans, are also morphisms of adjunctions.

Lemma 16. *For $w \in \{p, c\}$ let \mathbb{A} be an \mathcal{F} -category with w -limits of loose morphisms and tight pullbacks.*

- (1) *At a composable pair $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ the pullback projection $p_{g,f} : W_g W_f \rightarrow W_f$ is part of an adjunction of the form $(1, p_{g,f} \dashv r_{g,f}, \eta_{g,f})$ – unique in having the property that the pullback square $(q_{g,f}, q_f) : p_{g,f} \rightarrow p_f$*

$$\begin{array}{ccc} W_g W_f & \xrightarrow{q_{g,f}} & W_g \\ p_{g,f} \downarrow & & \downarrow p_g \\ W_f & \xrightarrow{q_f} & B \end{array}$$

is a morphism of adjunctions. When $w = p$ the 2-cell $\eta_{g,f}$ is invertible so that $(1, p_{g,f} \dashv r_{g,f}, \eta_{g,f})$ is an adjoint equivalence.

- (2) *The commuting square $(k_{g,f}, 1) : p_f p_{g,f} \rightarrow p_{gf}$*

$$\begin{array}{ccc} W_g W_f & \xrightarrow{k_{g,f}} & W_{gf} \\ p_{g,f} \downarrow & & \downarrow p_{gf} \\ W_f & & \\ p_f \downarrow & & \downarrow 1 \\ A & \xrightarrow{1} & A \end{array}$$

is a morphism of adjunctions.

Proof. (1) This is an instance of a general 2-categorical fact. Given a pullback square like left below

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ f_1 \downarrow & & \downarrow f_2 \\ C & \xrightarrow{s} & D \end{array} \quad \begin{array}{ccc} C & \xrightarrow{g_2 s} & B \\ 1 \downarrow & & \downarrow f_2 \\ C & \xrightarrow{s} & D \end{array} \quad \begin{array}{ccc} C & \xrightarrow{s} & D \\ g_1 \downarrow & & \downarrow g_2 \\ A & \xrightarrow{r} & B \end{array}$$

in any 2-category suppose that we have an adjunction $(1, f_2 \dashv g_2, \eta_2)$ – this pulls back to a unique adjunction $(1, f_1 \dashv g_1, \eta_1)$ of the same form with the property that $(r, s) : f_1 \rightarrow f_2$ is a morphism of adjunctions. To see this observe that the central square commutes and therefore induces a unique $g_1 : A \rightarrow C$ such that $f_1 g_1 = 1$ and such that the right square above commutes. By the 2-dimensional universal property of the pullback A there exists a unique 2-cell $\eta_1 : 1 \Rightarrow g_1 f_1$ such that $f_1 \eta_1 = 1$ and $r \eta_1 = \eta_2 r$. This gives one triangle equation for the adjunction $(1, f_1 \dashv g_1, \eta_1)$ – the other equation $\eta_1 g_1 = 1$ also follows from the universal property of the pullback. Now (r, s) commutes with both left and right adjoints and units by construction of η_1 ; as the counits are identities it commutes with these automatically so giving the claimed morphism of adjunctions. Note that if η_2 is invertible then, since the pullback projections are jointly conservative, so too is η_1 . This gives the case $w = p$.

- (2) To show that $(k_{g,f}, 1) : p_f p_{g,f} \rightarrow p_{gf}$ is a morphism of adjunctions it suffices, by Lemma 6, to show that the mate of this square is an identity. Now the counits of both adjunctions $p_{g,f} \dashv r_{g,f}$ and $p_f \dashv r_f$ are identities – thus the counit of the composite adjunction $p_f p_{g,f} \dashv r_{g,f} r_f$ is an identity so that the mate of $(k_{g,f}, 1)$ is simply

$$\begin{array}{ccccc}
 & W_g W_f & \xrightarrow{k_{g,f}} & W_{gf} & \xrightarrow{1} & W_{gf} \\
 & \downarrow p_f p_{g,f} & & \downarrow p_{gf} & \Downarrow \eta_{gf} & \\
 r_{g,f} r_f \nearrow & A & \xrightarrow{1} & A & \xrightarrow{1} & A & \nwarrow r_{gf}
 \end{array}$$

Now the 2-dimensional universal property of W_{gf} implies that the projections p_{gf} and q_{gf} jointly reflect identity 2-cells: see Lemma 3.1 of [12] in the case of the colax limit. Therefore it suffices to show that the composite of the above 2-cell with both p_{gf} and q_{gf} yields an identity. Now as the adjunction $p_{gf} \dashv r_{gf}$ has identity counit one of its triangle equations gives $p_{gf} \eta_{gf} = 1$; thus it remains to show $q_{gf} \eta_{gf} k_{g,f} r_{g,f} r_f = 1$. By (4.1) this equals $\lambda_{gf} k_{g,f} r_{g,f} r_f$ and by (4.4) we have $\lambda_{gf} k_{g,f} = (g \lambda_f p_{g,f}) \cdot (\lambda_g q_{g,f})$. Therefore it suffices to show that the 2-cells $(g \lambda_f p_{g,f}) r_{g,f} r_f$ and $(\lambda_g q_{g,f}) r_{g,f} r_f$ are identities separately. With regards the former we have that $\lambda_f p_{g,f} r_{g,f} r_f = \lambda_f r_f = 1$ where we first use that $p_{g,f} \dashv r_{g,f}$ has identity counit and then (4.1); for the other composite we have $(\lambda_g q_{g,f}) r_{g,f} r_f = \lambda_g r_g q_f r_f = 1$. The first equation holds since $(q_{g,f}, q_f)$ is a morphism of adjunctions by Part 1 and the second equation by (4.1). \square

5. ORTHOGONALITY

The main result of this section, Theorem 17, is the crucial result of the paper. Whilst our monadicity theorems of Section 6 follow easily from this theorem we note that both Theorem 17 and Corollary 18 are independent of the formalism of 2-monads.

5.1. Orthogonality and orthogonal decompositions. Before proving Theorem 17 recall from Section 4.4 our inability, in the pseudo-setting, to represent arbitrary 2-cells $\alpha : f \Rightarrow g \in \mathbb{A}(A, B)$ by span maps $P_f \rightarrow P_g \in \text{Span}(U\mathcal{A}_\tau)(A, B)$. To overcome this difficulty we use *cotensors with $\mathbf{2}$* . These are tight limits and so created by each of the forgetful \mathcal{F} -functors $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{C}$ for $w \in \{l, p, c\}$ (see Proposition 11); furthermore they are special instances of both colax limits and lax limits of loose morphisms (of identity morphisms) so that assuming their existence in the lax or colax settings adds nothing new. Given $A \in \mathbb{A}$ its cotensor with $\mathbf{2}$, A^2 , comes equipped with a triple (d, η, c) as below

$$\begin{array}{c}
 \begin{array}{ccc}
 & d & \\
 \curvearrowright & & \curvearrowright \\
 A^2 & \eta \Downarrow & A \\
 \curvearrowleft & & \curvearrowleft \\
 & c &
 \end{array}
 \quad
 \begin{array}{ccc}
 & r & \\
 \curvearrowright & & \curvearrowright \\
 B & \alpha \Downarrow & A \\
 \curvearrowleft & & \curvearrowleft \\
 & s &
 \end{array}
 \quad = \quad
 \begin{array}{ccc}
 & d & \\
 \curvearrowright & & \curvearrowright \\
 B & \rightsquigarrow \rightsquigarrow \rightsquigarrow t \rightsquigarrow \rightsquigarrow \rightsquigarrow & A^2 & \eta \Downarrow & A \\
 \curvearrowleft & & \curvearrowleft \\
 & c &
 \end{array}
 \end{array}$$

whose 1-dimensional universal property, depicted on the right, is that given any other 2-cell $\alpha : r \Rightarrow s$ in $\mathcal{A}_\lambda(B, A)$ there exists a unique loose morphism $t : B \rightsquigarrow A^2$

such that $dt = r, ct = s$ and $\eta t = \alpha$. This asserts that the induced functor $\mathcal{A}_\lambda(B, A^2) \cong \mathcal{A}_\lambda(B, A)^2$ is bijective on objects; the 2-dimensional universal property, not needed here, asserts that this functor is fully faithful. The \mathcal{F} -categorical aspect asserts that the projections d and c are tight and jointly detect tightness. Observe how cotensors with $\mathbf{2}$ allow us to decompose each 2-cell in \mathbb{A} as a composite of a 2-cell in \mathcal{A}_τ and a loose morphism. We will make use of this fact in the following theorem.

Theorem 17. *Let $w \in \{l, p, c\}$. Consider an \mathcal{F} -category \mathbb{A} with \overline{w} -limits of arrows, tight pullbacks and cotensors with $\mathbf{2}$. Then the inclusion of tight morphisms $j : \mathcal{A}_\tau \rightarrow \mathbb{A}$ is orthogonal to each w -doctrinal \mathcal{F} -functor.*

Proof. Consider a commuting square in $\mathcal{F}\text{-CAT}$

$$\begin{array}{ccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A} \\ R \downarrow & \swarrow K & \downarrow S \\ \mathbb{B} & \xrightarrow{W} & \mathbb{C} \end{array}$$

in which W is w -doctrinal. We must show there exists a unique diagonal filler K . We will consider only the case $w = l$. The pseudo-case is identical in form with adjunctions replaced by adjoint equivalences throughout. The case $w = c$ is formally dual to the lax case since \mathbb{A} satisfies the c -criteria of the theorem just when \mathbb{A}^{co} satisfies the l -criteria with, equally, W c -doctrinal just when W^{co} is l -doctrinal.

- (1) Before constructing the diagonal we fix some notation and make some observations about lifted adjunctions that will be repeatedly used in what follows. Given an adjunction of the form $(1, f \dashv g, \eta) \in \mathbb{A}$ with tight left adjoint f we obtain an adjunction $(1, Sf \dashv Sg, S\eta)$ in \mathbb{C} with $Sf = WRf$ since f is tight. As W is l -doctrinal this lifts uniquely along W to an adjunction in \mathbb{B} which we denote by $(1, Rf \dashv \overline{g}, \overline{\eta})$.

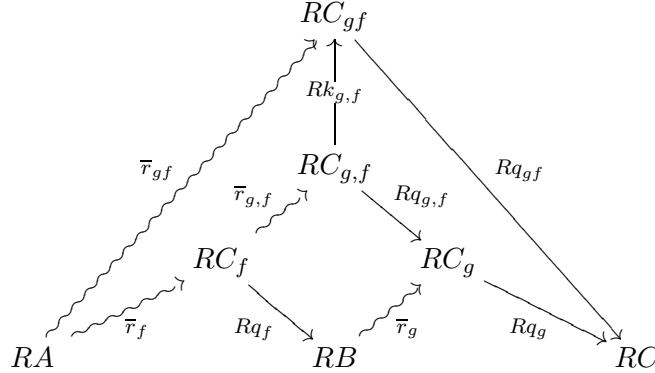
Next consider adjunctions $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in \mathbb{A} with tight left adjoints f_1 and f_2 and a tight morphism of adjunctions $(r, s) : f_1 \rightarrow f_2$ in \mathbb{A} as left below

$$\begin{array}{ccccc} A & \xrightarrow{r} & C & & RA & \xrightarrow{Rr} & RC & & RB & \xrightarrow{Rs} & RD \\ f_1 \downarrow & & \downarrow f_2 & & Rf_1 \downarrow & & \downarrow Rf_2 & & \overline{g_1} \downarrow & & \downarrow \overline{g_2} \\ B & \xrightarrow{s} & D & & RB & \xrightarrow{Rs} & RD & & RA & \xrightarrow{Rr} & RC \end{array}$$

Since \mathcal{F} -functors preserves morphisms of adjunctions $(Sr, Ss) : Sf_1 \rightarrow Sf_2$ is one in \mathbb{C} . Now the tight map $(Rr, Rs) : Rf_1 \rightarrow Rf_2$ of left adjoints in \mathbb{B} has image under W the morphism of adjunctions $(Sr, Ss) : Sf_1 \rightarrow Sf_2$. Because W is l -doctrinal it follows that $(Rr, Rs) : Rf_1 \rightarrow Rf_2$ is a morphism between the lifted adjunctions in \mathbb{B} so that, in particular, we have a commuting square of right adjoints $(Rs, Rr) : \overline{g_1} \rightarrow \overline{g_2}$.

Finally consider a composable pair of tight left adjoints $f_1 : A \rightarrow B$ and $f_2 : B \rightarrow C$ with associated adjunctions $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in

- \mathbb{A} . We can form the adjunction $(1, Rf_1 \dashv \overline{g_1}, \overline{\eta_1})$ and $(1, Rf_2 \dashv \overline{g_2}, \overline{\eta_2})$ in \mathbb{B} and compose these to obtain an adjunction $Rf_2 Rf_1 \dashv \overline{g_1 g_2}$ in \mathbb{B} of the same form. It is clear that this is a lifting of the composite adjunction $S(f_2 f_1) \dashv S(g_1 g_2)$ so that, by uniqueness of liftings, the adjunctions $Rf_2 Rf_1 \dashv \overline{g_1} \cdot \overline{g_2}$ and $R(f_2 f_1) \dashv \overline{g_1 g_2}$ coincide.
- (2) Now to begin constructing K observe that for the left triangle to commute we must define $KA = RA$ for each $A \in \mathbb{A}$.
- (3) Given $f : A \rightsquigarrow B \in \mathbb{A}$ we recall from (4.1) its factorisation as $q_f r_f : A \rightsquigarrow C_f \rightarrow B$ where $(1, p_f \dashv r_f, \eta_f)$. Since p_f is tight we have the lifted adjunction $(1, Rp_f \dashv \overline{r_f}, \overline{\eta_f})$ in \mathbb{B} living over $(1, Sp_f \dashv Sr_f, S\eta_f)$ and define $Kf : RA \rightsquigarrow RB$ to be the composite $Rq_f \overline{r_f} : RA \rightsquigarrow RC_f \rightarrow RB$.
- (4) Observe that $WKf = WRq_f W\overline{r_f} = WRq_f Sr_f = S(q_f r_f) = Sf$ as required for the right triangle to commute. To see that K extends R observe that if $f : A \rightarrow B$ is tight then, by (4.2), r_f is tight too, so that we have an adjunction $(1, Rp_f \dashv Rr_f, R\eta_f)$ living over $(1, WRp_f \dashv Sr_f, \eta_f)$; thus $Rr_f = \overline{r_f}$ and $Kf = Rr_f Rp_f = Rf$. Thus K coincides with R on tight morphisms.
- (5) As K agrees with R on tight morphisms we already know that it preserves identity 1-cells. To see that it preserves composition of 1-cells it will suffice to show that all of the regions of the diagram below commute.



The rightmost triangle of four arrows certainly commutes as it is the image of a commutative diagram in \mathcal{A}_τ , from (4.4), under R . To see that the central square commutes recall from Lemma 16.1 that we have a tight morphism of adjunctions $(q_{g,f}, q_f) : p_{g,f} \rightarrow p_g$ in \mathbb{A} . By Part 1 $(Rq_{g,f}, Rq_f) : Rp_{g,f} \rightarrow p_{g,f}$ is a morphism of adjunctions in \mathbb{B} : now commutativity of the central square just asserts commutativity with right adjoints. With regards the left triangle of four arrows recall from Lemma 16.2 the tight morphism of adjunctions in \mathbb{A} given by $(k_{g,f}, 1) : p_f p_{g,f} \rightarrow p_{gf}$. By Part 1 again $(Rk_{g,f}, 1) : R(p_f p_{g,f}) \rightarrow Rp_{gf}$ is a morphism of the lifted adjunctions so that, in particular, we have a commuting square of right adjoints $(1, Rk_{g,f}) : \overline{r}_{g,f} \overline{r_f} \rightarrow \overline{r}_{gf}$ in \mathbb{B} . Moreover by Part 1 again we know that $\overline{r}_{g,f} \overline{r_f} = \overline{r}_{g,f} \overline{r_f}$ so that we have the map of right adjoints $(1, Rk_{g,f}) : \overline{r}_{g,f} \overline{r_f} \rightarrow \overline{r}_{gf}$ – this exhibits the desired commutativity of the triangle.

- (6) Each 2-cell $\alpha : f \Rightarrow g \in \mathbb{A}(A, B)$

$$\begin{array}{c}
 \begin{array}{ccc}
 & f & \\
 & \curvearrowright & \\
 A & \alpha \Psi & B \\
 & \curvearrowleft & \\
 & g &
 \end{array}
 & = &
 \begin{array}{ccc}
 & d & \\
 & \curvearrowright & \\
 A & \rightsquigarrow h & B^2 \\
 & \curvearrowleft & \\
 & c &
 \end{array}
 \end{array}$$

factors uniquely through the cotensor B^2 via a loose morphism h . Thus define

$$K\alpha = R\eta Kh : Kf = KdKh = RdKh \Rightarrow RcKh = KcKh = Kg$$

We then have $WK\alpha = WR\eta WKh = S\eta Sh = S\alpha$ so that $WK = S$ on 2-cells. To see that K extends R on 2-cells observe that if f and g are tight then, by the \mathcal{F} -categorical universal property of B^2 , h is also tight so that we have $K\alpha = R\eta Kh = R\eta Rh = R(\eta h) = R\alpha$.

- (7) Because W is locally faithful the functoriality of K on 2-cells trivially follows from the functoriality of $WK = S$ on 2-cells. Thus K is an \mathcal{F} -functor.
- (8) As for uniqueness of K suppose that L were a second filler for the square. Then we would need $Lf = L(q_f r_f) = Lq_f Lr_f = Rq_f Lr_f$. Since $WL = S$ we would need that the image of the adjunction $(1, p_f \dashv r_f, \eta_f)$ under L were a lifting of its image under WL , $(1, Sp_f \dashv Sr_f, S\eta_f)$. But by uniqueness of liftings we would then have $(1, Lp_f \dashv Lr_f, L\eta_f) = (1, Rp_f \dashv \overline{r_f}, \overline{\eta_f})$ whence $Lr_f = \overline{r_f}$ and $Lf = Lq_f \overline{r_f} = Kf$. Thus L and K must coincide on loose morphisms. Now decomposing a 2-cell $\alpha = \eta h$ using cotensors with $\mathbf{2}$ and using that L coincides with K on loose morphisms and with R on \mathcal{A}_τ we deduce that $L\alpha = R\eta Lh = R\eta Kh = K\alpha$ so that L and K agree on 2-cells. \square

Let us remark that Theorem 17 remains true when $w = l$ or $w = c$ even if we remove the assumption that w -doctrinal \mathcal{F} -functors are locally faithful. The only place we used this assumption was in establishing the functoriality of the diagonal K on 2-cells – this can alternatively be established by carefully analysing the representation of 2-cells in \mathbb{A} by morphisms of tight spans (see Section 4.4). Because local faithfulness is easy to verify in practice, considerably shortens the proof when $w = l$ and $w = c$ and is apparently required when $w = p$, we assumed it across the board even though it is somewhat unnatural.

As an immediate consequence of Theorem 17 we have:

Corollary 18. *Let $w \in \{l, p, c\}$. Consider an \mathcal{F} -functor $W : \mathbb{A} \rightarrow \mathcal{B}$ to a 2-category and suppose that \mathbb{A} has \overline{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$, and that W is w -doctrinal. Then the decomposition in \mathcal{F} -CAT*

$$\mathcal{A}_\tau \xrightarrow{j} \mathbb{A} \xrightarrow{W} \mathcal{B}$$

of the 2-functor $W_\tau : \mathcal{A}_\tau \rightarrow \mathcal{B}$ is an orthogonal $({}^\perp w\text{-Doct}, w\text{-Doct})$ -decomposition.

Since orthogonal decompositions are essentially unique, this asserts that an \mathcal{F} -category satisfying some completeness properties and sitting over a base 2-category in a certain way – such as MonCat_w or T-Alg_w over Cat – is *uniquely determined* by how its tight part sits over the base. This is the core idea behind our monadicity results of Section 6.

5.2. A note on alternative hypotheses. There are other hypotheses on \mathbb{A} which also ensure that $j : \mathcal{A}_\tau \rightarrow \mathbb{A}$ belongs to ${}^\perp w\text{-Doct}$. Focusing on the lax case the idea is that we want to represent each loose morphism $f : A \rightsquigarrow B$ by a tight span as on the left below

$$\begin{array}{ccc} & C & \\ p \swarrow & & \searrow q \\ A & & B \end{array} \qquad \begin{array}{ccc} & QA & \\ p_A \swarrow & & \searrow f' \\ A & & B \end{array}$$

in which p has a loose right adjoint r satisfying $f = qr$. Of course this representation must be *well behaved* and we must be able to *extend it in some way to 2-cells*. One other set of hypotheses on \mathbb{A} which allows us to do this is as follows. Say that \mathbb{A} admits *tightening* if $j : \mathcal{A}_\tau \rightarrow \mathcal{A}_\lambda$ has a left 2-adjoint Q ; then the isomorphism $\mathcal{A}_\tau(QA, B) \cong \mathcal{A}_\lambda(A, B)$ exhibits QA as a *loose morphism classifier* with each loose morphism $f : A \rightsquigarrow B$ factoring uniquely through the unit $p_A : A \rightsquigarrow QA$ as some tight $f' : QA \rightarrow B$. The counit $q_A : QA \rightarrow A$ is a tight map satisfying the triangle equation $q_A p_A = 1_A$. If in fact we have an adjunction $(1, q_A \dashv p_A, \eta_A)$ then each f can be represented by a tight span as on the right above – under such circumstances it can be easily shown that $j : \mathcal{A}_\tau \rightarrow \mathbb{A} \in {}^\perp l\text{-Doct}$. Now if \mathbb{A} admits tightening and colax limits of loose morphisms then we always have such adjunctions $(1, q_A \dashv p_A, \eta_A)$ – via a simple \mathcal{F} -categorical abstraction of Lemma 2.5 of [15] – and so orthogonality. More generally *if \mathbb{A} admits tightening and \overline{w} -limits of loose morphisms then $j : \mathcal{A}_\tau \rightarrow \mathbb{A}$ is orthogonal to each w -doctrinal \mathcal{F} -functor*. Of course these hypotheses are strong: it is not easy to check whether a given \mathcal{F} -category admits tightening.

6. MONADICITY

In this section we give our monadicity theorems. We begin by extending Eilenberg-Moore comparison 2-functors to \mathcal{F} -functors and show, in Theorem 20, that these comparison \mathcal{F} -functors are natural in w , in the sense of Diagram 2 of the introduction. Our main monadicity result is Theorem 21.

6.1. Extending the Eilenberg-Moore comparison.

Theorem 19. *Let $W : \mathbb{A} \rightarrow \mathcal{B}$ be an \mathcal{F} -functor to a 2-category whose tight part $W_\tau : \mathcal{A}_\tau \rightarrow \mathcal{B}$ has a left adjoint and consider the induced Eilenberg-Moore comparison 2-functor E left below.*

$$\begin{array}{ccc} \mathcal{A}_\tau & \xrightarrow{E} & \mathbf{T}\text{-Alg}_s \\ & \searrow W_\tau & \swarrow U_s \\ & \mathcal{B} & \end{array} \qquad \begin{array}{ccccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A} & \xrightarrow{W} & \mathcal{B} \\ E \downarrow & & \downarrow E_w & & \uparrow U \\ \mathbf{T}\text{-Alg}_s & \xrightarrow{j_w} & \mathbf{T}\text{-Alg}_w & & \end{array}$$

Let $w \in \{l, p, c\}$ and suppose that \mathbb{A} admits \overline{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$. Then $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s$ admits a unique extension to an \mathcal{F} -functor $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-Alg}_w$ over \mathcal{B} , as depicted on the right above.

Proof. The commutativity of the outside of the right diagram just says that $U_s E = W_\tau$. By Corollary 9 and Theorem 17 we know that $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{B}$ is w -doctrinal and that $j : \mathcal{A}_\tau \rightarrow \mathbb{A}$ is orthogonal to each w -doctrinal \mathcal{F} -functor, in particular U . Therefore there exists a unique \mathcal{F} -functor $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-Alg}_w$ satisfying the depicted equations $E_w j = j_w E$ and $U E_w = W$. These respectively assert that E_w extends E and lives over the base \mathcal{B} . \square

In order to understand the naturality in w of the above Eilenberg-Moore extensions $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-Alg}_w$ it will be convenient to briefly consider \mathcal{F}_2 -categories. An \mathcal{F}_2 -category consists of a 2-category equipped with three kinds of morphism: tight, loose and very loose, all satisfying the expected axioms. For instance we have the \mathcal{F}_2 -category of monoidal categories, strict, strong and lax monoidal functors; likewise of algebras together with strict, pseudo and lax morphisms for a 2-monad. One presentation of an \mathcal{F}_2 -category is as a triple on the left below

$$\mathcal{A}_\tau \xrightarrow{j} \mathcal{A}_\lambda \xrightarrow{j} \mathcal{A}_\phi \qquad \mathbb{A}_{\tau,\lambda} \xrightarrow{j} \mathbb{A}_{\tau,\phi}$$

in which each inclusion is the identity on objects, faithful and locally fully faithful. Thus an \mathcal{F}_2 -category has three associated \mathcal{F} -categories $\mathbb{A}_{\tau,\lambda}$, $\mathbb{A}_{\lambda,\phi}$ and $\mathbb{A}_{\tau,\phi}$, and is determined by the first and third of these together with the inclusion \mathcal{F} -functor, right above, which views tight and loose morphisms as tight and very loose respectively.

We commonly encounter \mathcal{F}_2 -categories sitting over a base 2-category as on the left below. See how monoidal categories, strict, strong and lax monoidal functors sit over \mathbf{Cat} for instance.

$$\begin{array}{ccccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathcal{A}_\lambda & \xrightarrow{j} & \mathcal{A}_\phi \\ & \searrow W_\tau & \downarrow W_\lambda & \swarrow W_\phi & \\ & & \mathcal{B} & & \end{array} \qquad \begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} \\ & \searrow W_{\tau,\lambda} & \swarrow W_{\tau,\phi} \\ & & \mathcal{B} \end{array}$$

To give such a diagram is equally to give a commutative triangle in $\mathcal{F}\text{-CAT}$ as on the right above – here the \mathcal{F} -functors $W_{\tau,\lambda}$ and $W_{\tau,\phi}$ agree as W_τ on tight parts, and have loose parts W_λ and W_ϕ respectively.

Theorem 20. *Let $w \in \{l, c\}$. Consider an \mathcal{F}_2 -category over a 2-category as below*

$$\begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} \\ & \searrow W_{\tau,\lambda} & \swarrow W_{\tau,\phi} \\ & & \mathcal{B} \end{array}$$

Suppose that W_τ admits a left adjoint and that $\mathbb{A}_{\tau,\lambda}$ and $\mathbb{A}_{\tau,\phi}$ satisfy the (p/w) variants of the completeness criteria of Theorem 19 so that the Eilenberg-Moore comparison 2-functor $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s$ extends uniquely to \mathcal{F} -functors $E_p : \mathbb{A}_{\tau,\lambda} \rightarrow \mathbf{T}\text{-Alg}_p$ and $E_w : \mathbb{A}_{\tau,\phi} \rightarrow \mathbf{T}\text{-Alg}_w$ over the base \mathcal{B} .

When all of this holds the square

$$\begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} \\ E_p \downarrow & & \downarrow E_w \\ \mathbf{T}\text{-}\mathbf{Alg}_p & \xrightarrow{j} & \mathbf{T}\text{-}\mathbf{Alg}_w \end{array}$$

commutes.

Proof. Consider the diagram left below

$$\begin{array}{ccccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} \\ & & E_p \downarrow & & \downarrow E_w \\ & & \mathbf{T}\text{-}\mathbf{Alg}_p & \xrightarrow{j} & \mathbf{T}\text{-}\mathbf{Alg}_w \xrightarrow{U_w} \mathcal{B} \end{array} \quad \begin{array}{ccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A}_{\tau,\lambda} \\ jE \downarrow & \swarrow \text{dotted} & \downarrow W_{\tau,\lambda} \\ \mathbf{T}\text{-}\mathbf{Alg}_w & \xrightarrow{U_w} & \mathcal{B} \end{array}$$

Since both E_p and E_w agree as $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_s$ on tight morphisms both paths of the square coincide upon precomposition with $j : \mathcal{A}_\tau \rightarrow \mathbb{A}_{\tau,\lambda}$ as the composite $jE : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_s \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_w$. Because both Eilenberg-Moore extensions E_p and E_w lie over the base \mathcal{B} we find that postcomposing both paths of the square with U_w yields the common composite $W_{\tau,\lambda} : \mathbb{A}_{\tau,\lambda} \rightarrow \mathcal{B}$. Therefore both paths of the square are diagonal fillers for the square on the right above. Now by Corollary 9 we know that $U_w : \mathbf{T}\text{-}\mathbf{Alg}_w \rightarrow \mathcal{B}$ is p -doctrinal. But by Theorem 17 $j : \mathcal{A}_\tau \rightarrow \mathbb{A}_{\tau,\lambda}$ is orthogonal to each p -doctrinal \mathcal{F} -functor so that both paths coincide as the unique filler. \square

6.2. 2-categorical monadicity. We now turn to monadicity. Recall that a 2-functor with a left adjoint is, by definition, monadic if the induced Eilenberg-Moore comparison is a 2-equivalence and strictly monadic if the comparison is an isomorphism.

Theorem 21. *Let $W : \mathbb{A} \rightarrow \mathcal{B}$ be an \mathcal{F} -functor to a 2-category \mathcal{B} . Let $w \in \{l, p, c\}$ and suppose that*

- (1) $W_\tau : \mathcal{A}_\tau \rightarrow \mathcal{B}$ is monadic.
- (2) \mathbb{A} admits \overline{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$.
- (3) \mathcal{B} admits \overline{w} -limits of arrows, pullbacks and cotensors with $\mathbf{2}$.
- (4) W is w -doctrinal (it suffices that W satisfies w -doctrinal adjunction, is locally faithful and reflects identity 2-cells).

Then the 2-equivalence $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_s$ extends uniquely to an equivalence of \mathcal{F} -categories $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_w$ over \mathcal{B} . Moreover if W_τ is strictly monadic then $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_w$ is an isomorphism of \mathcal{F} -categories.

Proof. As in Theorem 19 we have our extension $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-Alg}_w$, unique in filling the square

$$\begin{array}{ccc}
 \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A} \\
 E \downarrow & \nearrow E_w & \downarrow W \\
 \mathbf{T}\text{-Alg}_s & & \mathbf{T}\text{-Alg}_w \\
 j_w \downarrow & \nearrow U & \downarrow \\
 \mathbf{T}\text{-Alg}_w & \xrightarrow{U} & \mathcal{B}
 \end{array}$$

Recall that this filler exists because U is w -doctrinal and j orthogonal to such \mathcal{F} -functors.

We begin by proving the theorem in the case of strict monadicity and deduce the general case from that – so suppose that $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s$ is an isomorphism of 2-categories. By Proposition 11 $U : \mathbf{T}\text{-Alg}_w \rightarrow \mathcal{B}$ creates \overline{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$ so that $\mathbf{T}\text{-Alg}_w$ admits these limits. Now Theorem 17 implies that the inclusion $j_w : \mathbf{T}\text{-Alg}_s \rightarrow \mathbf{T}\text{-Alg}_w$ is orthogonal to each w -doctrinal \mathcal{F} -functor; since $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s$ is an isomorphism $j_w E : \mathcal{A} \rightarrow \mathbf{T}\text{-Alg}_w$ is also orthogonal to w -doctrinal \mathcal{F} -functors. Now W is w -doctrinal by assumption so that the two outer paths of the square are orthogonal decompositions of a common \mathcal{F} -functor. Therefore the unique filler $E_w : \mathbb{A} \rightarrow \mathbf{T}\text{-Alg}_w$ is an isomorphism.

Suppose now that $E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s$ is only an equivalence of 2-categories. The problem now is that E will no longer be orthogonal to w -doctrinal \mathcal{F} -functors. We will rectify this by factoring the composite 2-functor $j_w E : \mathcal{A}_\tau \rightarrow \mathbf{T}\text{-Alg}_s \rightarrow \mathbf{T}\text{-Alg}_w$ as the identity on objects followed by 2-fully faithful through a 2-category $\overline{\mathcal{A}}_\lambda$ as in the commutative square below

$$\begin{array}{ccc}
 \mathcal{A}_\tau & \xrightarrow{\overline{j}} & \overline{\mathcal{A}}_\lambda \\
 E \downarrow & & \downarrow K \\
 \mathbf{T}\text{-Alg}_s & \xrightarrow{j_w} & \mathbf{T}\text{-Alg}_w
 \end{array}$$

Then K is 2-fully faithful by construction but also essentially surjective on objects since each of E , \overline{j} and j_w are; as such K is a 2-equivalence. Moreover the composite $K\overline{j} = j_w E$ is both faithful and locally fully faithful since both j_w and E are, whilst K is 2-fully faithful. Therefore \overline{j} is faithful and locally fully faithful too and since it is identity on objects by construction it is the inclusion of an \mathcal{F} -category $\overline{\mathbb{A}} : \mathcal{A}_\tau \rightarrow \overline{\mathcal{A}}_\lambda$. It now follows that the above commutative square, whose vertical legs are 2-equivalences, exhibits $L = (E, K) : \overline{\mathbb{A}} \rightarrow \mathbf{T}\text{-Alg}_w$ as an equivalence of \mathcal{F} -categories.

Consider the diagram defining the extension E_w again, now drawn on the left

below

$$\begin{array}{ccc}
 \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A} \\
 \bar{j} \downarrow & \nearrow \overline{E} & \downarrow W \\
 \overline{\mathbb{A}} & & \mathbb{B} \\
 L \downarrow & \nearrow E_w & \\
 \mathbf{T}\text{-}\mathbf{Alg}_w & \xrightarrow{U} & \mathbf{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A} \\
 \bar{j} \downarrow & \nearrow \overline{E} & \downarrow W \\
 \overline{\mathbb{A}} & \xrightarrow{L} \mathbf{T}\text{-}\mathbf{Alg}_w & \xrightarrow{U} \mathbf{B}
 \end{array}$$

with the left leg rewritten as the composite $L \circ \bar{j}$. Since L is \mathcal{F} -fully faithful (2-fully faithful on tight and loose parts) it is orthogonal to each bijective on objects \mathcal{F} -functor, in particular j , so that we obtain a unique diagonal filler $\overline{E} : \mathbb{A} \rightarrow \overline{\mathbb{A}}$ making the two leftmost triangles commute.

Our goal is to show that E_w is an equivalence of \mathcal{F} -categories - but since L is an equivalence this is, by 2 out of 3, equivalently to show that \overline{E} is an equivalence of \mathcal{F} -categories. Now consider the square on the right. The bottom leg is w -doctrinal as both of its components are, L by Proposition 10. Since the \mathcal{F} -category $\overline{\mathbb{A}}$ is equivalent to $\mathbf{T}\text{-}\mathbf{Alg}_w$ it has the same completeness properties, so that, using Theorem 17 again, the left leg is orthogonal to each w -doctrinal \mathcal{F} -functor. Therefore the right commutative square consists of two orthogonal decompositions of a common \mathcal{F} -functor and we conclude that \overline{E} is an isomorphism. \square

Note that although Theorem 21, in each of its variants, only asks that \mathbb{A} admits certain limits it follows that W creates those limits: for $U : \mathbf{T}\text{-}\mathbf{Alg}_w \rightarrow \mathbf{B}$ does so by Proposition 11 and $E : \mathbb{A} \rightarrow \mathbf{T}\text{-}\mathbf{Alg}_w$ is an equivalence of \mathcal{F} -categories. In applying the theorem this is often useful in that it tells us how these limits must be constructed in \mathbb{A} .

6.3. \mathcal{F} -categorical monadicity. Our focus has been upon 2-monads but indeed the above results extend in a routine way to cover \mathcal{F} -categorical monadicity too. Let us briefly explain how this goes. An \mathcal{F} -monad [15] is a monad in the 2-category $\mathcal{F}\text{-CAT}$ and so consists of an \mathcal{F} -functor $T : \mathbb{A} \rightarrow \mathbb{A}$ and two \mathcal{F} -natural transformations satisfying the usual equations. 2-monads are just \mathcal{F} -monads on 2-categories viewed as \mathcal{F} -categories. An \mathcal{F} -monad T induces 2-monads T_τ and T_λ and so we have strict T_τ and strict T_λ -algebra morphisms as in the two leftmost diagrams below.

$$\begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 a \downarrow & & \downarrow b \\
 A & \xrightarrow{f} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 TA & \overset{Tf}{\rightsquigarrow} & TB \\
 a \downarrow & & \downarrow b \\
 A & \rightsquigarrow_f & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 TA & \overset{Tf}{\rightsquigarrow} & TB \\
 a \downarrow & \Downarrow \bar{f} & \downarrow b \\
 A & \rightsquigarrow_f & B
 \end{array}$$

These are the tight and loose morphisms of the Eilenberg-Moore \mathcal{F} -category which is denoted by $\mathbf{T}\text{-}\mathbf{Alg}_s$. We also have pseudo, lax and colax T_λ -morphisms – a lax T_λ -morphism is drawn above. These are the loose morphisms of the \mathcal{F} -categories

$\mathbf{T}\text{-Alg}_w$ whose tight morphisms are the strict T_τ -algebra maps; now for each $w \in \{l, p, c\}$ we have the \mathcal{F}_2 -category whose tight and loose morphisms are the strict T_τ and T_λ -morphisms and whose very loose morphisms are the w - T_λ -morphisms; as captured by the inclusion \mathcal{F} -functor $j_w : \mathbf{T}\text{-Alg}_s \rightarrow \mathbf{T}\text{-Alg}_w$. We have the evident forgetful \mathcal{F} -functors $U_s : \mathbf{T}\text{-Alg}_s \rightarrow \mathbb{A}$ and $U_w : \mathbf{T}\text{-Alg}_w \rightarrow \mathbb{A}$ commuting with j_w over the base. The key point for our applications is that these have the same properties as in the 2-monads case: namely U_s creates all limits, U_w creates cotensors with $\mathbf{2}$, tight pullbacks and \bar{w} -limits of loose morphisms (this follows from Theorem 5.13 of [15]) and U is w -doctrinal. With these facts in place we can give our monadicity theorem for \mathcal{F} -monads – we leave it to the reader to formulate the naturality of the Eilenberg-Moore extensions, which can be done using \mathcal{F}_3 -categories.

Theorem 22. *Consider an \mathcal{F}_2 -category $\mathcal{A}_\tau \rightarrow \mathcal{A}_\lambda \rightarrow \mathcal{A}_\phi$ over an \mathcal{F} -category \mathbb{B} as below*

$$\begin{array}{ccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} \\ & \searrow W & \swarrow V \\ & \mathbb{B} & \end{array}$$

(this means that $W_\tau = V_\tau$). Suppose that W has a left \mathcal{F} -adjoint so that we have the Eilenberg-Moore comparison \mathcal{F} -functor $E : \mathbb{A}_{\tau,\lambda} \rightarrow \mathbf{T}\text{-Alg}_s$ over the base \mathbb{B} . Now let $w \in \{l, p, c\}$ and suppose that both $\mathbb{A}_{\tau,\lambda}$ and $\mathbb{A}_{\tau,\phi}$ admit \bar{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$. Then there exists a unique \mathcal{F} -functor $E_w : \mathbb{A}_{\tau,\phi} \rightarrow \mathbf{T}\text{-Alg}_w$ extending E and living over the base, as depicted by the following everywhere commutative diagram

$$\begin{array}{ccccc} \mathbb{A}_{\tau,\lambda} & \xrightarrow{j} & \mathbb{A}_{\tau,\phi} & & \\ \downarrow E & & \downarrow E_w & \searrow V & \\ \mathbf{T}\text{-Alg}_s & \xrightarrow{j_w} & \mathbf{T}\text{-Alg}_w & \nearrow U_w & \mathbb{B} \end{array}$$

If W is \mathcal{F} -monadic, \mathbb{B} admits \bar{w} -limits of loose morphisms, tight pullbacks and cotensors with $\mathbf{2}$ and V is w -doctrinal then E_w is an equivalence of \mathcal{F} -categories, and an isomorphism of \mathcal{F} -categories whenever W is strictly monadic.

Proof. The outside of the diagram clearly commutes – since $Vj = W$ and $U_w j_w = U_s$ this just amounts to the fact that the Eilenberg-Moore comparison E satisfies $U_s E = W$. Now $U_w : \mathbf{T}\text{-Alg}_w \rightarrow \mathbb{B}$ is w -doctrinal so that if we can show that the inclusion $j : \mathbb{A}_{\tau,\lambda} \rightarrow \mathbb{A}_{\tau,\phi}$ belongs to ${}^\perp w\text{-Doct}$ then we will obtain E_w as the unique filler. Now have a commutative triangle of inclusions

$$\begin{array}{ccc} \mathcal{A}_\tau & \xrightarrow{j} & \mathbb{A}_{\tau,\lambda} \\ & \searrow j & \downarrow j \\ & & \mathbb{A}_{\tau,\phi} \end{array}$$

in which the two j 's moving from left to right belong to ${}^\perp w\text{-Doct}$ by Theorem 17; thus by 2 out of 3 $j : \mathbb{A}_{\tau,\lambda} \rightarrow \mathbb{A}_{\tau,\phi} \in {}^\perp w\text{-Doct}$. As such we obtain the Eilenberg-Moore extension E_w as the unique filler. The remainder of the proof is a straightforward modification of the proof of Theorem 21. \square

7. EXAMPLES AND APPLICATIONS

We now turn to examples. We begin by completing our running example of monoidal categories. Of course it is well known that monoidal categories, and each flavour of morphism between them, can be described using 2-monads – see Section 5.5 of [14] for an argument via colimit presentations – although this has not previously been established by application of a monadicity theorem. We then turn to more complex examples. Our final example, in 7.3, is new and typical of the kind of result which cannot be established using techniques, such as colimit presentations, that require explicit knowledge of a 2-monad.

7.1. Monoidal categories. Let us focus, as usual, on the lax monoidal functors of MonCat_l . From Example 5 we know that $V : \text{MonCat}_l \rightarrow \text{Cat}$ satisfies l -doctrinal adjunction. It is clearly locally faithful and reflects identity 2-cells. From Example 13 we know that $V : \text{MonCat}_l \rightarrow \text{Cat}$ creates colax limits of loose morphisms – recall that these include cotensors with $\mathbf{2}$ as a simple special case. Furthermore it is easy to see that $V_s : \text{MonCat}_s \rightarrow \text{Cat}$ creates pullbacks and that the inclusion $j : \text{MonCat}_s \rightarrow \text{MonCat}_l$ preserves them; thus MonCat_l admits tight pullbacks.

Therefore to apply Theorem 21 and establish monadicity it remains to verify that the 2-functor $V_s : \text{MonCat}_s \rightarrow \text{Cat}$ is monadic. That V_s strictly creates V_s -absolute coequalisers (in the enriched sense) is true by essentially the same argument given for groups in 6.8 of [17]; by Beck's theorem in the enriched setting [3] it follows that V_s is strictly monadic so long as it has a left 2-adjoint. We have already seen that V_s creates cotensors with $\mathbf{2}$; Proposition 3.1 of [2] then ensures that V_s admits a left 2-adjoint just when its underlying functor $(V_s)_0$ admits a left adjoint. Since this functor creates all limits it suffices to show that $(V_s)_0$ satisfies the solution set condition: to this end it suffices to show that given a small category A each functor $F : A \rightarrow C = U\overline{C}$ to a monoidal category factors as $ME : A \rightarrow B \rightarrow C$ with B monoidal, M strict monoidal and the cardinality of the set of morphisms $\text{Mor}(B)$ bounded by that of $\text{Mor}(A)$; here B will be the monoidal subcategory of C generated by the image of F and M the inclusion of this monoidal subcategory. For the induced 2-monad T on Cat we now conclude, by Theorem 21, that the isomorphism of 2-categories $E : \text{MonCat}_s \rightarrow \text{T-Alg}_s$ over Cat extends uniquely to an isomorphism of \mathcal{F} -categories $E_l : \text{MonCat}_l \rightarrow \text{T-Alg}_l$ over Cat . Likewise one can verify, in an entirely similar way, that $V : \text{MonCat}_w \rightarrow \text{Cat}$ satisfies the conditions of Theorem 21 in the cases $w \in \{p, c\}$ so that we have isomorphisms $E_w : \text{MonCat}_w \rightarrow \text{T-Alg}_w$ over Cat for each $w \in \{l, p, c\}$, with Theorem 20 then ensuring that these isomorphisms are natural in $p \leq l$ and $p \leq c$ in the sense of Diagram 2 of the introduction.

7.2. Categories with structure and variants. Of course there was nothing special about our taking monoidal categories in the preceding section – the same

arguments can be used to establish monadicity of categories with any kind of algebraically specified structure and their various flavours of morphisms: categories with chosen limits of some kind for instance, distributive categories and so forth. All of these cases are well known to be monadic using colimit presentations of 2-monads, although it requires substantial and laborious calculation to use that theory to establish monadicity in the detailed manner above – such a detailed treatment using colimit presentations, one expressed in terms of isomorphisms of 2-categories or \mathcal{F} -categories, will not be found in the literature. Colimit presentations are one of the two standard techniques for understanding the monadicity of weaker kinds of morphisms; the other is direct calculation with a 2-monad known to exist. As a representative example of this technique consider a small 2-category \mathcal{J} and the forgetful 2-functor $U : [\mathcal{J}, \text{Cat}] \rightarrow [\text{ob}\mathcal{J}, \text{Cat}]$ which restricts presheaves to families along the inclusion $\text{ob}\mathcal{J} \rightarrow \mathcal{J}$. U has a left 2-adjoint F given by left Kan extension and is strictly monadic by Beck's theorem; moreover the induced 2-monad $T = UF$ admits a simple pointwise description: at $X \in [\text{ob}\mathcal{J}, \text{Cat}]$ we have $TX(j) = \Sigma_i \mathcal{J}(i, j) \times Xi$. Using this formula (as in [2]) one directly calculates that T -pseudomorphisms bijectively correspond to pseudonatural transformations and so on, eventually deducing an isomorphism $Ps(\mathcal{J}, \text{Cat}) \rightarrow \text{T-Alg}_p$. It is in such cases, more specifically when T is not so simple, that our results have most value. An example of this kind was given in [9]. Given a complete and cocomplete symmetric monoidal closed category \mathcal{V} the authors consider the 2-category V-Cat of small \mathcal{V} -categories and, for a small class of weights Φ , the 2-category $\Phi\text{-Colim}$ (we will write $\Phi\text{-Colim}_s$) of \mathcal{V} -categories with *chosen* Φ -weighted colimits, \mathcal{V} -functors preserving those colimits strictly and \mathcal{V} -natural transformations, all living over V-Cat via a forgetful 2-functor $U_s : \Phi\text{-Colim}_s \rightarrow \text{V-Cat}$. One also has \mathcal{V} -functors preserving colimits in the usual, up to isomorphism, sense: these are the loose morphisms of the \mathcal{F} -category $\Phi\text{-Colim}_p : \Phi\text{-Colim}_s \rightarrow \Phi\text{-Colim}_p$ which sits over V-Cat via a forgetful \mathcal{F} -functor $U : \Phi\text{-Colim}_p \rightarrow \text{V-Cat}$. The authors show that $U_s : \Phi\text{-Colim}_s \rightarrow \text{V-Cat}$ is strictly monadic and then, by calculating directly with the induced 2-monad T , show that one obtains an isomorphism of 2-categories $U : \Phi\text{-Colim}_p \rightarrow \text{T-Alg}_p$. Let us show how this can be deduced from Theorem 21. Firstly observe that U satisfies p -doctrinal adjunction: this follows from the fact that any equivalence of \mathcal{V} -categories preserves colimits. Again since U is locally fully faithful it is certainly locally faithful and reflects identity 2-cells. It is not hard to see that $\Phi\text{-Colim}_p$ admits pseudo-limits of loose morphisms – indeed this is shown in Section 5 of [9] – and that it admits pullbacks of tight morphisms and cotensors with **2**. It then follows immediately from Theorem 21 that the isomorphism of 2-categories $E : \Phi\text{-Colim}_s \rightarrow \text{T-Alg}_s$ extends uniquely to an isomorphism of \mathcal{F} -categories $E : \Phi\text{-Colim}_p \rightarrow \text{T-Alg}_p$ over V-Cat .

Furthermore, because left adjoints preserve colimits, $U : \Phi\text{-Colim}_p \rightarrow \text{V-Cat}$ has the additional property of satisfying c -doctrinal adjunction; since it is isomorphic to T-Alg_p it also admits, by Theorem 2.6 of [2], lax limits of loose morphisms. Therefore Theorem 21 ensures that the composite $\Phi\text{-Colim}_p \rightarrow \text{T-Alg}_p \rightarrow \text{T-Alg}_c$ is also an isomorphism, thus explaining why the colax and pseudo T -morphisms coincide as those \mathcal{V} -functors which preserve Φ -colimits. Again one easily applies Theorem 21 to show that T-Alg_l is isomorphic to the \mathcal{F} -category $\Phi\text{-Colim}_l$ whose loose morphisms are arbitrary \mathcal{V} -functors.

7.3. In a monoidal 2-category. In a monoidal category \mathcal{C} one can consider the category of monoids $\text{Mon}(\mathcal{C})$ or of commutative monoids. If the forgetful functor $U : \text{Mon}(\mathcal{C}) \rightarrow \mathcal{C}$ has a left adjoint then Beck's theorem can be applied, with no further information, to show that U is monadic. For our final example we study the analogous situation in the context of a monoidal 2-category \mathcal{C} : here one can consider monoids, pseudomonoids (generalising monoidal categories), braided pseudomonoids and so forth – we consider only the simplest case of monoids because, in the absence of a suitable graphical calculus, it is difficult to encode diagrams compactly. We have the 2-category of monoids, strict monoid morphisms and monoid transformations $\text{Mon}(\mathcal{C})_s$ and a forgetful 2-functor $U_s : \text{Mon}(\mathcal{C})_s \rightarrow \mathcal{C}$ and, just as before, the enriched version of Beck's theorem [3] can be applied to show that if U_s has a left 2-adjoint it is monadic; however we now also have (lax/pseudo/colax)-morphisms of monoids and we would like to understand that these too are monadic in the appropriate sense – this is the content, under completeness conditions on the base, of the present example.

By a monoidal 2-category \mathcal{C} we will mean a monoidal \mathcal{V} -category where $\mathcal{V} = \text{Cat}$: this satisfies the same axioms as a monoidal category with the exception that the tensor product, the associator and the other data involved are now 2-functorial and 2-natural. In working with \mathcal{C} we will write as though it were strict monoidal: this is justified in the theorem that follows. A monoid in \mathcal{C} is just a monoid in the usual sense. Given monoids (X, m_X, i_X) and (Y, m_Y, i_Y) a lax monoid map $(f, \bar{f}, f_0) : (X, m_X, i_X) \rightarrow (Y, m_Y, i_Y)$ consists of an arrow $f : X \rightarrow Y$ and 2-cells as below

$$\begin{array}{ccc} X^2 & \xrightarrow{f^2} & Y^2 \\ m_X \downarrow & \bar{f} \Downarrow & \downarrow m_Y \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} & I & \\ i_X \swarrow & f_0 \Downarrow & \searrow i_Y \\ X & \xrightarrow{f} & Y \end{array}$$

such that the equation

$$\begin{array}{ccccc} X^3 & \xrightarrow{1ff} & XY^2 & \xrightarrow{f11} & Y^3 \\ 1m_X \downarrow & 1\bar{f} \Downarrow & \downarrow 1m_Y & & \downarrow 1m_Y \\ X^2 & \xrightarrow{1f} & XY & \xrightarrow{f1} & Y^2 \\ m_X \downarrow & \bar{f} \Downarrow & & \downarrow m_Y & \\ X & \xrightarrow{f} & & & Y \end{array} = \begin{array}{ccccc} X^3 & \xrightarrow{11f} & X^2Y & \xrightarrow{ff1} & Y^3 \\ m_X 1 \downarrow & m_X 1 \downarrow & \bar{f} 1 \Downarrow & & \downarrow m_Y 1 \\ X^2 & \xrightarrow{1f} & XY & \xrightarrow{f1} & Y^2 \\ m_X \downarrow & \bar{f} \Downarrow & & \downarrow m_Y & \\ X & \xrightarrow{f} & & & Y \end{array}$$

holds and such that both composite 2-cells

$$\begin{array}{ccccc} X & \xrightarrow{1} & X & \xrightarrow{f} & Y \\ i_X \downarrow & 1f_0 \Downarrow & i_Y \downarrow & & \downarrow i_Y \\ X^2 & \xrightarrow{1f} & XY & \xrightarrow{f1} & Y^2 \\ m_X \downarrow & 1\bar{f} \Downarrow & & \downarrow m_Y & \\ X & \xrightarrow{f} & & & Y \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{1} & Y \\ i_X 1 \downarrow & i_X 1 \downarrow & f_0 1 \Downarrow & & \downarrow i_Y 1 \\ X^2 & \xrightarrow{1f} & XY & \xrightarrow{f1} & Y^2 \\ m_X \downarrow & 1\bar{f} \Downarrow & & \downarrow m_Y & \\ X & \xrightarrow{f} & & & Y \end{array}$$

are identities. A monoid transformation $\alpha : (f, \bar{f}, f_0) \Rightarrow (g, \bar{g}, g_0)$ is a 2-cell $\alpha : f \Rightarrow g$ satisfying the equations

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & f^2 & \\
 X^2 & \xrightarrow{\alpha^2 \Downarrow} & Y^2 \\
 m_X \downarrow & & \downarrow m_Y \\
 & g^2 & \\
 & \Downarrow \bar{g} & \\
 X & \xrightarrow{g} & Y
 \end{array}
 & = &
 \begin{array}{ccc}
 & f^2 & \\
 X^2 & \xrightarrow{\bar{f} \Downarrow} & Y^2 \\
 m_X \downarrow & & \downarrow m_Y \\
 & f & \\
 & \alpha \Downarrow & \\
 X & \xrightarrow{g} & Y
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 & I & \\
 i_X \swarrow & & \searrow i_Y \\
 X & \xrightarrow{f_0 \Downarrow} & Y \\
 & \alpha \Downarrow & \\
 X & \xrightarrow{g} & Y
 \end{array}
 =
 \begin{array}{ccc}
 & I & \\
 i_X \swarrow & & \searrow i_Y \\
 X & \xrightarrow{g_0 \Downarrow} & Y \\
 & g & \\
 X & \xrightarrow{g} & Y
 \end{array}$$

These are the 2-cells of the \mathcal{F} -category $\mathbf{Mon}(\mathcal{C})_l : \mathbf{Mon}(\mathcal{C})_s \rightarrow \mathbf{Mon}(\mathcal{C})_l$ of monoids, strict and lax monoid morphisms which sits over \mathcal{C} via a forgetful \mathcal{F} -functor $U : \mathbf{Mon}(\mathcal{C})_l \rightarrow \mathcal{C}$. Likewise we have pseudo and colax monoid morphisms and forgetful \mathcal{F} -functors $U : \mathbf{Mon}(\mathcal{C})_w \rightarrow \mathcal{C}$ for each $w \in \{l, p, c\}$.

Theorem 23. *Let \mathcal{C} be a finitely complete monoidal 2-category and suppose that $U_s : \mathbf{Mon}(\mathcal{C})_s \rightarrow \mathcal{C}$ has a left 2-adjoint. Then for each $w \in \{l, p, c\}$ we have isomorphisms of \mathcal{F} -categories $\mathbf{Mon}(\mathcal{C})_w \rightarrow \mathbf{T-Alg}_w$ over \mathcal{C} and these are natural in w .*

Proof. Let us begin by showing that it suffices to suppose \mathcal{C} to be strict monoidal. A straightforward extension of the usual argument for monoidal categories shows that \mathcal{C} is equivalent to a strict monoidal 2-category \mathcal{D} via a strong monoidal 2-equivalence $E : \mathcal{C} \rightarrow \mathcal{D}$. Such an equivalence naturally lifts to a 2-equivalence $E_* : \mathbf{Mon}(\mathcal{C})_w \rightarrow \mathbf{Mon}(\mathcal{D})_w$ for each w and so induces a commuting square of \mathcal{F} -categories and \mathcal{F} -functors

$$\begin{array}{ccc}
 \mathbf{Mon}(\mathcal{C})_w & \xrightarrow{E_*} & \mathbf{Mon}(\mathcal{D})_w \\
 U_C \downarrow & & \downarrow U_D \\
 \mathcal{C} & \xrightarrow{E} & \mathcal{D}
 \end{array}$$

with both horizontal legs equivalences of \mathcal{F} -categories. Now to apply Theorem 21 we must show that $U_C : \mathbf{Mon}(\mathcal{C})_w \rightarrow \mathcal{C}$ is w -doctrinal and that $\mathbf{Mon}(\mathcal{C})_w$ has \bar{w} -limits of loose morphisms, tight pullbacks and cotensors with **2**. So suppose that U_D and $\mathbf{Mon}(\mathcal{D})_w$ have these properties and let us deduce from these the corresponding properties for \mathcal{C} . Certainly if U_D were w -doctrinal then U_C would be too; for both horizontal legs, being equivalences, are w -doctrinal and such \mathcal{F} -functors, being defined by lifting properties (as in Section 3.3), are closed under 2 out of 3. Likewise any limits existing in $\mathbf{Mon}(\mathcal{D})_w$ exist in the \mathcal{F} -equivalent $\mathbf{Mon}(\mathcal{C})_w$; therefore it suffices to suppose that \mathcal{C} is strict monoidal.

Now certainly $U : \mathbf{Mon}(\mathcal{C})_l \rightarrow \mathcal{C}$ is locally faithful and reflects identity 2-cells. Moreover given a strict monoid map $(f, \bar{f}, f_0) : X \rightarrow Y$ and adjunction $(\epsilon, f \dashv$

$g, \eta) \in \mathcal{C}$ taking mates gives 2-cells

$$\begin{array}{ccc}
 Y^2 & \xrightarrow{g^2} & X^2 \\
 \downarrow 1 & \swarrow \epsilon^2 & \downarrow m_X \\
 Y^2 & \xleftarrow{f^2} & X \\
 m_Y \downarrow & \swarrow f & \downarrow 1 \\
 Y & \xrightarrow{g} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xrightarrow{i_X} & X \\
 i_Y \downarrow & \swarrow f & \downarrow 1 \\
 Y & \xrightarrow{g} & X
 \end{array}$$

It is straightforward to see, by cancelling mates, that these give g the structure of a lax monoid map (g, \bar{g}, g_0) , with respect to which $(\epsilon, f \dashv (g, \bar{g}, g_0), \eta)$ is an adjunction in $\text{Mon}(\mathcal{C})_l$; uniqueness of this lifted adjunction again follows from Proposition 3.1 so that U satisfies l -doctrinal adjunction.

We will show that $U : \text{Mon}(\mathcal{C})_l \rightarrow \mathcal{C}$ creates tight pullbacks and colax limits of loose morphisms – tight pullbacks are straightforward so we consider only the second case. Consider a lax monoid map $(f, \bar{f}, f_0) : X \rightarrow Y$ and the colax limit C of f in \mathcal{C} with limiting cone as below

$$\begin{array}{ccc}
 & C & \\
 p \swarrow & \lambda \Downarrow & \searrow q \\
 X & \xrightarrow{f} & Y
 \end{array}$$

By the universal property of this cone the composite cone left below induces a unique map $m_C : CC \rightarrow C$ such that $pm_C = m_X p^2$, $qm_C = m_Y q^2$ and such that the left equation below holds. Likewise we obtain a unique $i_C : I \rightarrow C$ such that $pi_C = i_X$, $qi_C = i_Y$ and satisfying $\lambda i_C = f_0$ as on the right below.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & C^2 & \\
 p^2 \swarrow & \lambda^2 \Downarrow & \searrow q^2 \\
 X^2 & \xrightarrow{ff} & Y^2 \\
 m_X \downarrow & \bar{f} \Downarrow & \downarrow m_Y \\
 X & \xrightarrow{f} & Y
 \end{array}
 & = &
 \begin{array}{ccc}
 & C^2 & \\
 p^2 \swarrow & m_C \downarrow & \searrow q^2 \\
 X^2 & \xrightarrow{p} C \xrightarrow{q} & Y^2 \\
 m_X \downarrow & \lambda \Downarrow & \downarrow m_Y \\
 X & \xrightarrow{f} & Y
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 & I & \\
 i_X \swarrow & f_0 \Downarrow & \searrow i_Y \\
 X & \xrightarrow{f} & Y
 \end{array}
 =
 \begin{array}{ccc}
 & I & \\
 i_X \swarrow & i_C \downarrow & \searrow i_Y \\
 X & \xrightarrow{p} C \xrightarrow{q} & Y \\
 m_X \downarrow & \lambda \Downarrow & \downarrow m_Y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

If we can show (C, m_C, i_C) to be a monoid then these combined equations will assert exactly that p and q are strict monoid maps and $\lambda : q \Rightarrow (f, \bar{f}, f_0).p$ a monoid transformation. To show that $m_C(m_C 1) = m_C(1 m_C) : C^3 \rightarrow C$ amounts to showing that both paths coincide upon postcomposition with the components p , q and λ of the universal cone. We have that $pm_C(m_C 1) = m_X p^2(m_C 1) = m_X(m_X 1)p^3 = m_X(1 m_X)p^3 = m_X p^2(1 m_C) = pm_C(1 m_C)$ and similarly for q so that associativity of m_C will follow if we can show that both paths coincide upon

straightforward and left to the reader – that p and q detect strict monoid morphisms is a consequence of the fact that they jointly detect identity 2-cells in \mathcal{C} . Now if $U_s : \text{Mon}(\mathcal{C})_s \rightarrow \mathcal{C}$ has a left adjoint it is automatically strictly monadic by the enriched version of Beck’s monadicity theorem [3]. Therefore, using the above, Theorem 21 asserts that the isomorphism of 2-categories $E : \text{Mon}(\mathcal{C})_s \rightarrow \text{T-Alg}_s$ over \mathcal{C} extends uniquely to an isomorphism of \mathcal{F} -categories $E_l : \text{Mon}(\mathcal{C})_l \rightarrow \text{T-Alg}_l$ over \mathcal{C} . In a similar way one verifies the conditions of Theorem 21 when $w \in \{p, c\}$ to obtain isomorphisms of \mathcal{F} -categories $E_w : \text{Mon}(\mathcal{C})_w \rightarrow \text{T-Alg}_w$ over \mathcal{C} for each w ; by Theorem 20 these isomorphisms are natural in w . \square

7.4. A non-example. All of the examples we have seen are of the strictly monadic variety and indeed this is the case whenever one studies structured objects over some same base 2-category. Now Theorem 21 is general enough to cover ordinary monadicity – up to equivalence of \mathcal{F} -categories – but in fact there exist situations of a weaker kind. Here is one such case. Let $\text{Cat}_f \subset \text{Cat}$ be a full sub 2-category of Cat whose objects form a skeleton of the finitely presentable categories (the finitely presentable objects in Cat) and let $[\text{Cat}, \text{Cat}]_f \subset [\text{Cat}, \text{Cat}]$ be the full sub 2-category consisting of those endo 2-functors preserving filtered colimits: this is the tight part of the \mathcal{F} -category $\mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f : [\text{Cat}, \text{Cat}]_f \rightarrow \mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f$ whose loose morphisms are pseudonatural transformations. Likewise we have an \mathcal{F} -category $\mathbb{P}\text{s}(\text{Cat}_f, \text{Cat}) : [\text{Cat}_f, \text{Cat}] \rightarrow \mathbb{P}\text{s}(\text{Cat}_f, \text{Cat})$ and now restriction along the inclusion $\text{Cat}_f \rightarrow \text{Cat}$ induces a forgetful \mathcal{F} -functor $R : \mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f \rightarrow \mathbb{P}\text{s}(\text{Cat}_f, \text{Cat})$. Further restricting along the inclusion $\text{obCat}_f \rightarrow \text{Cat}_f$ gives a commuting triangle

$$\begin{array}{ccc} \mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f & \xrightarrow{R} & \mathbb{P}\text{s}(\text{Cat}_f, \text{Cat}) \\ & \searrow SR \quad \swarrow S & \\ & [\text{obCat}_f, \text{Cat}] & \end{array}$$

The composite $S_\tau R_\tau : [\text{Cat}, \text{Cat}]_f \rightarrow [\text{obCat}_f, \text{Cat}]$ is monadic though not strictly so: for the induced 2-monad T we have T-Alg_p isomorphic to $\mathbb{P}\text{s}(\text{Cat}_f, \text{Cat})$ with $R_\tau : [\text{Cat}, \text{Cat}]_f \rightarrow [\text{Cat}_f, \text{Cat}]$ the Eilenberg-Moore comparison 2-functor – that this is a 2-equivalence follows from Cat ’s being locally finitely presentable as a 2-category. Whilst R_τ is a 2-equivalence the 2-functor $R_\lambda : \mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f \rightarrow \mathbb{P}\text{s}(\text{Cat}_f, \text{Cat})$ is not: indeed $\mathbb{P}\text{s}(\text{Cat}_f, \text{Cat})$ is locally small whereas $\mathbb{P}\text{s}(\text{Cat}, \text{Cat})_f$ is not. Yet R_λ turns out to be a biequivalence and R the uniquely induced \mathcal{F} -functor to the \mathcal{F} -category of algebras: we hope to treat this weaker setting and give applications of it in a short further paper.

REFERENCES

- [1] BECK, J. Triples, algebras and cohomology. *Rep. Theory Appl. Categ.* 2 (2003), 1–59.
- [2] BLACKWELL, R., KELLY, G. M., AND POWER, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra* 59, 1 (1989), 1–41.
- [3] DUBUC, E. J. *Kan extensions in enriched category theory*, vol. 145 of *Lecture Notes in Mathematics*. Springer, 1970.

- [4] GRANDIS, M., AND THOLEN, W. Natural weak factorization systems. *Archivum Mathematicum* 42, 4 (2006), 397–408.
- [5] GRAY, J. The meeting of the Midwest Category Seminar in Zurich. *Lecture Notes in Mathematics* 195 (1971), 248–255.
- [6] KELLY, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.
- [7] KELLY, G. M. On clubs and doctrines. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 181–256.
- [8] KELLY, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.
- [9] KELLY, G. M., AND LACK, S. On the monadicity of categories with chosen colimits. *Theory and Applications of Categories* 7, 7 (2000), 148–170.
- [10] KELLY, G. M., AND POWER, A. J. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra* 89, 1–2 (1993), 163–179.
- [11] KELLY, G. M., AND STREET, R. Review of the elements of 2-categories. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 75–103.
- [12] LACK, S. Limits for lax morphisms. *Appl. Categ. Structures* 13, 3 (2002), 189–203.
- [13] LACK, S. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures* 7, 2 (2007), 229–260.
- [14] LACK, S. A 2-categories companion. In *Towards higher categories*, vol. 152 of *IMA Vol. Math. Appl.* Springer, 2010, pp. 105–191.
- [15] LACK, S., AND SHULMAN, M. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics* 229, 1 (2011), 294–356.
- [16] LE CREURER, I., MARMOLEJO, F., AND VITALE, E. Beck’s theorem for pseudomonads. *Journal of Pure and Applied Algebra* 173, 3 (2002), 293–313.
- [17] MAC LANE, S. *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics*. Springer, 1971.
- [18] SHULMAN, M. Comparing composites of left and right derived functors. *New York Journal of Mathematics* 17 (2011), 75–125.
- [19] STREET, R. Fibrations and Yoneda’s lemma in a 2-category. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 104–133.

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